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PRIMARY ALGEBRA

COMPLETE EDITION

J. W. MacDONALD

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MacDonald's Mathematical Series.

PRIMARY ALGEBRA.

THE TEACHER'S GUIDE.

BY

J. W. MacDONALD.

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MACDONALD'S PRIMARY ALGEBRA.

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PREFACE.

THE purpose of this book is made sufficiently evident by its name. For grammar school use it covers the work of about one year, but can be finished in much less time in a high school or academy.

It is made in two parts: A Teacher's Guide, and A Student's Manual. The former contains all the theory, explanations, definitions, etc.; the latter, lesson by lesson, furnishes examples for class drill, and for the student's home work, and is the only part that the pupils should have.

This arrangement enables the teacher to take up each topic fresh with his pupils; to arouse their interest and stimulate inquiry; to develop the principles for them in logical order, and let them deduce from practice their own definitions and rules.

Hence the pupils will not become confused and discouraged over lifeless text-book directions, introduced wrong end first; and the teacher's work will not be merely explaining to pupils the meaning of text-book instructions, of which they have previously learned just enough to take away their appetite.

Besides the dual form of the book, its other main distinctive features are: (1) the scientific explanation and treatment of negative quantities; (2) the theory of algebraic quantities in general; (3) the theory and law of signs in addition, subtraction, multiplication, and division, and the way of performing the first two of these operations; (4) the explanation of the effect of the parenthesis; and (5) the treatment of equations.

It is the setting forth of these principles on a basis somewhat different from what other algebras teach, that has necessitated making the first seventeen lessons of the Teacher's Guide so full. It is not intended that the teacher should learn this verbatim, or use it without change. It should serve merely as a series of hints, to be extended whenever circumstances demand fuller treatment, or to be abbreviated when intelligent results can be obtained in less time than that allowed here. The teacher's success will depend largely on the individuality and ingenuity displayed in following out the general plan. Above all, the teacher must furnish the life and arouse the enthusiasm, without which any teaching is well-nigh profitless.

Neither is it intended to have the pupils, in this first course, learn formal discussions of all the theories. This would, at so early a stage, cost too much time.

But the theories should be repeatedly talked over and reviewed, and many examples should be given to illustrate them. In this way they will gradually ripen in the pupil's mind, and in a more advanced course he will himself be able to construct complete scientific discussions of them, at much less cost of time.

The same may be said of other explanations and of definitions. They should not be forced too early. They should follow familiarity with the process, not precede it. I once saw a teacher teaching a class to square a polynomial of four terms. He had the pupils square $a + b + c + d$, and then from this single experience set them to explaining the process. He thus spent nearly the whole recitation period, and yet only a few succeeded; and in the one example given them to do as a test, all failed. Had he, after squaring the above polynomial, used it for a formula, and had a number of other examples done by it, the pupils would have acquired facility in doing the work, and most of them at the end of the period could have explained the steps. An immense amount of time is wasted in teaching mathematics by doing what should be last, first, as this teacher did.

There are minor points of difference between this book and others, one of them the use of the word "eliminate" instead of "transpose." When transpose is used, it is in its common meaning, as, $6 = x$ transposed gives $x = 6$, not $-x = -6$. Pupils should be taught that transposing the members of an equation does *not* change the signs. If $7 = x$, then, of course, $x = 7$.

Some of the lessons in the Student's Manual are starred. Answers to some of the problems in these will be given at the end of the book. In other cases, where answers are not given, the pupils should prove their answers. This will be a good practice in arithmetical work. One of the objects in all departments of education should be to cultivate in the pupils self-reliance; and much is lost in this respect where the answers to problems in their mathematical work are within easy reach.

My obligations are due to Mr. George A. Bacon for material assistance in correcting the proof-sheets of the book, and for valuable suggestions both as to the matter and the manner of presentation.

J. W. MACDONALD.

STONEHAM, *June* 1, 1894.

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TEACHER'S GUIDE

TO

PRIMARY ALGEBRA.

Base what the pupils are learning on what they have already learned.

LESSON I.

How a letter may be used in solving certain Problems — Coefficients.

NOTE. As a rule call up the pupils individually, but some questions may be addressed to the class. Use judgment in this.

TEACHER. Scholars, what branch of mathematics have you been studying heretofore?

PUPILS. Arithmetic.

T. In arithmetic how have you expressed magnitudes, such as quantity, value, etc.?

P. By numbers.

T. How have you expressed the numbers? (Pupils hesitate[1].) If you wanted to express the number *seven*, how have you written it? John, put it on the board. (John goes to the board, and writes 7.) What do you call that? P. Seven.

---

[1] In cases like this give the pupils time to think, but do not waste time.

T. Yes, *but I wanted the general name* for it. For example, what do you call these symbols? (Writing *a*, *b*, *c*, etc.) P. Letters.

T. What do you call these? (writing 1, 2, 3, 4, etc.)
P. Figures.

T. What do we use these for?
P. To express numbers.

T. Have you ever expressed numbers in any other way? P. We have spelled them out.

T. Yes. Have you ever expressed quantities in any other way still? (The pupils agree that they have not.) Well, scholars, in higher mathematics and in algebra, which you are now beginning to study, the more common way of expressing magnitudes is not by figures, but by letters; and you may be surprised to learn that problems have been solved by means of letters that would never have been solved by means of figures alone. There are also many problems that can be solved more easily by the use of letters than by the use of figures. Now let us see how this can be done. Take this problem:

Two times a certain number, plus three times the same number, is seventy-five; what is the number?

Now I will write it on the board, and you may copy it on your paper:—

2 times a certain number
3 times the same number.

What will adding these together give us? (Repeating the above.)

P. Five times the number.

T. And what does it equal?

P. Seventy-five.

T. Then let us put it in writing: —

5 times a certain number = 75.

What must be the number?

P. Fifteen.

T. Then finish by writing: —

The number = 15.

This is undoubtedly the number, but let us prove it. If fifteen is the number, what would twice the number be?

P. Thirty.

T. And three times the number?

P. Forty-five.

T. Adding them?

P. Seventy-five.

T. Right. Now let us try it another way. (Repeating the problem.) Let x be the number. Now, I want you to think of x as being the number; that is, as being the answer. If x is the number, what is twice the number?

P. $2x$. (Teacher writes it.)

T. And three times the number?

P. $3x$. (Teacher writes it under the $2x$.)

T. How many x is $2x$ and $3x$?

P. $5x$. (Teacher writes it.)

T. And what does $5x$ equal?

P. 75.

T. Let us put it into a good form. (Writing on the board and pupils copying.)

$$\begin{array}{l} 2\,x \\ \underline{3\,x} \\ 5\,x = 75 \\ \quad x = 15. \end{array}$$

What have we saved by using x?

P. We have saved a good deal of writing.

T. And consequently time. Let us try another. I have in mind a certain number; tell me what it is. (The pupils look puzzled.) Tell me right off.

P. We don't know.

T. You mean you don't know how to express it in figures, but you have learned another way. What is the number? (Some of the pupils timidly suggest x.) Yes, that's it. Why did you hesitate? Now I am going to tell you something about the number, and see if it will prove. Four times the number plus five times the number equals ninety-nine. Now tell me promptly, if x is the number, what will four times the number be?

P. $4\,x$.

T. And 5 times?

P. $5\,x$.

T. And their sum?

P. $9\,x$.

T. Right; and what does $9\,x$ equal?

P. 99.

T. Then what must x be?

P. 11.

T. Good. Now let us put it into form.

Note. The teacher then has the pupils carefully write it out as follows: —

Let x be the number.

Then, $4x + 5x = 9x = 99$

$$x = 11.$$

Or,

$$\begin{array}{l} 4x \\ \underline{5x} \\ 9x = 99 \\ \quad x = 11. \end{array}$$

T. Now, in this same way, working as carefully and rapidly as you can, solve the following problems: —

Problem. 1. Five times a certain number and eight times the same number together equal seventy-eight. What is the number?

Note. See that these problems are all put into a neat and compact form.

2. I have in mind a certain number; once the number, four times the number, and seven times the number is one hundred and fifty-six. See if you can tell me the number?

T. (After looking over some of the work) I see that you all have written "once the number" in this way, $1x$. Now, the practice in algebra is to avoid every unnecessary mark. Let us see if there isn't something in that expression that can be omitted. How many fingers, including the thumb, have you on one hand?

P. Five.

T. You didn't say one five. Please write the number. (Pupils write 5.) That is good. You didn't write, 1 times 5 or 1×5. If, however, I should ask you how many fives of fingers you had on both hands what would you say?

P. Two fives.

T. And write it thus: 2 times 5, or 2×5; and so on for any other number of fives except once 5. Now, what does x alone mean? How many x?

P. One x.

T. Any need of writing the 1?

P. No, sir.

T. Then, always omit the coefficient of a letter when it is 1. Algebra tries to avoid everything needless. I see by your looks that I have used a word that you are not familiar with. It was this one (writing *coefficient* on the board). This is a very convenient word to be familiar with. It is one of the time-saving devices that algebra abounds in. Take, for example, the quantity 5×7; I might want to refer to the five, and I might call it "the factor by which 7 would have to be multiplied to produce 35," or I might call 7 "the factor by which 5 would have to be multiplied to produce 35." So in $6\,x$, which means 6 times x, I might call 6 the factor by which x is multiplied to produce $6\,x$; or I might call x the factor by which 6 is multiplied to produce $6\,x$. We are obliged frequently to refer to one of the factors in such cases, and such an expression as "the factor that x is multiplied by," etc., would be awkward and burdensome. We are able to avoid it by using the word "*coefficient.*" To show how much we save, let me ask you, in $9\,x$ what is the factor by which x is multiplied?

P. Nine.

T. In $9\,x$ what is the coefficient of x?

P. Nine.

T. What is the factor by which the 9 is multiplied?

P. x.

T. What is the coefficient of 9? P. x.

T. Do you see the advantage of using the word *coefficient?*

P. Yes, sir.

Note. The teacher can run rapidly through some of the expressions on the board, as $4x$, x, $5x$, etc., calling for the coefficient of one of the factors, both literal and numerical. Never call for the "coefficient in a term," as this means nothing. There may be a good many of them.

T. Now, scholars, work out for me a few examples.

1. A string 84 inches long is cut so that one piece is six times as long as the other. How long is each piece?

Now tell me right off what is the length of one of the pieces? Don't stop to compute. P. x.

T. Which one, the shorter or the longer?

P. The shorter.

T. Then, what would be the length of the other, which is six times as long?

P. $6x$.

T. Adding the lengths of the two pieces together what does it give you?

P. $7x$.

T. What is $7x$ equal to?

P. 84 inches.

T. Write it out in full form and finish it. (The pupils do so, and when the answer is called for it is given as 12 inches.) Why, you have only found the length of one of the pieces, how long is the other?

Note. The answer is made complete. The pupils are cautioned to notice what the problem gives and calls for, and whatever time is left can be spent on the following problems:—

1. A farmer sold 182 bushels of apples to four men; the first took a certain number of bushels, another 6 times as many, another 4 times as many, and the last 3 times as many; how many did each take?

2. John has a certain number of marbles, Henry has twice as many, and Willie as many as both John and Henry. The whole number of marbles was 96; how many had each?

Examples in Lesson I. of the Scholars' Manual may be brought in, worked out, at the next recitation.

Aim first to get the minds of the pupils into an inquiring mood; next stimulate them to try to discover the principles and processes for themselves; if they fail to do this, they will nevertheless have gotten their minds into an attitude to comprehend what has to be told them.

LESSON II.

Adding and Subtracting Positive Literal Quantities — Terms.

TEACHER. One problem by way of review. A and B set out from places 187 miles apart, and travel towards each other. A travels 8 miles an hour, and B 9 miles per hour; how long before they will meet?

First carefully note the question. What was it?

PUPILS. "How long before they meet?"

T. Well, tell me how long, promptly.

P. x.

T. x what? Your answer is vague, and vagueness is one of the worst faults of language. If you should ask me how long it took me to do some piece of work, and I should reply "five," would you consider the question answered?

P. No, sir; I meant x hours.

T. That is better. Always try to make your statement clear the first time. Who can tell us what to do next? (As only one or two hands go up, the teacher continues.) Perhaps we had better illustrate the situation by a drawing. John, go to the board and locate the two places, which we will call A and B. (Pupil does so.) Draw a line to show the road. (Pupil does so.) How long does the problem give that line?

P. 187 miles.

T. Thank you for putting in the word miles; I was afraid you would only say 187. About where would they meet if A goes 8 miles an hour and B goes 9?

P. About there (pointing to a place on the line).

T. Very well; mark it M, capital M, please, because we are using the small letters for quantities. (The pupil does so.) How far is it from A to M?

P. I don't know.

T. Can any one tell me how far it is from A to M, or from B to M? (As no hands are raised, the teacher continues.) Well, that is strange! Let me see if you can do this. C travels 3 hours at the rate of 4 miles an hour; how far did he go?

P. 12 miles.

T. That is right. How did you find it out?

P. I multiplied the rate by the time.

T. Why don't you do the same in this case?

P. We don't know the time.

T. Why, John told you; x hours. (Hands begin to go up.)

P. A travels $8x$ miles, and B $9x$ miles.

T. Then form the equation[1] and finish the problem. (After this is done.) That will be enough of this for to-day. I now want to take up something else. You have perhaps noticed that in the problems we have been solving, certain quantities are given you, and from them you are required to find other quantities not given, generally called unknown quantities. (The teacher illustrates this by some of the problems already given.) You have noticed further that the letter we have used to designate this unknown quantity has been x. y is also used, or any of the last letters of the alphabet. But in pure algebra we also use letters to designate quantities that would be given in such problems as we have been doing, but to show that they represent known quantities we use the first letters of the alphabet, as a, b, c, etc. I will not explain to you now what the advantage of using letters in this way to express quantities is; that will appear in good time. What I want you to understand for the present is, that if I say a marbles or b marbles, I mean a certain quantity of marbles, just as if I said 6 marbles or 7 marbles; that c means a quantity of anything just as much as 12, 13, or 20 does.[2] But first tell me, what have you learned in arithmetic to do with figures? P. I learned to add them.

T. Don't you mean that you learned to add numbers expressed by figures. P. Yes, sir.

---

[1] Whenever a technical term is used in this book without explanation, keep using it till some pupil asks what it means. Then explain it.

[2] The difference is that arithmetic deals with quantities as measured by units, that is, as numbers; algebra deals with them as quantities *en masse*, not divided into units.

T. Well, that is something, but I wish you had learned more. If I should tell you that Mr. A had five horses and bought three more, you could tell me how many he then had. But if I should tell you he had five horses and lost three, you couldn't tell me how many he had left. I should have to ask some one else.

P. Yes, I could; he would have two left.

T. Why, you didn't tell me you had learned to subtract. All you said was that you had learned to add. Now tell me just what you have learned to do.

P. I have learned to add, subtract, multiply, and divide numbers.

T. That is better. I thought it strange if, after studying arithmetic all these years, you had only learned to add. But have any of you ever learned any other process in arithmetic besides these four? (A few hands come up slowly.)

P. I learned interest.

T. Do you mean that you learned interest, or that you learned to compute interest?

P. That I learned to compute interest.

T. Now, scholars, let me ask you to be careful to use language that expresses what you mean. I find some of you inclined to say one thing when you mean another. In mathematics, language that isn't exact is very poor language indeed. But, to return to our subject, you say that you have learned to compute interest; will you show us how computing interest differs from adding, subtracting, multiplying, and dividing? Go to the board and work out this problem: What is the interest on $300, for four years, at six per cent?

NOTE. **Insist upon having the example worked out, whether the pupil claims to see the point or not, and in this way emphasize the fact that there is no other process in arithmetic besides the four just mentioned.**

T. Then, scholars, if we are to use letters to express quantities, what four things must we learn to do with them?

P. We must learn to add, subtract, multiply, and divide with them.

T. Then, let us set about it. You have already partly learned to add. 4 *a* and 5 *a* are how many?

P. 9 *a*.

T. 6 *a* and 5 *a*? P. 11 *a*.

T. 3 *a* from 20 *a*? P. 17 *a*.

T. 12 *a* and 3 *a*, less 5 *a*? P. 10 *a*.

T. How many would that be if *a* equals 3?

P. 30.

T. John had 5 *a* marbles and he got 6 *a* more; how many did he then have? P. 11 *a*.

T. He lost 7 *a*; how many had he left? P. 4 *a*.

T. John had *a* marbles and Henry had *a* marbles; how many had both? P. 2 *a*.

T. John had *a* marbles and Henry had *b* marbles; how many had they both? (The scholars look puzzled.) Why don't you add them?

P. We don't know how.

T. Is the sum 2 *a*? P. No, sir.

T. 2 *b*? P. No, sir.

T. Then, what is it? (No one can tell.) Well, scholars, if we are to use letters to express numbers, we must find a way to add *a* and *b*, as well as *a* and *a*. Can any one think of a way to write *a* and *b* to indicate that they are added together? (Some one suggests *ab*.) Well, that might do if all mathematicians agreed to it. But you know there are three other processes, and mathematicians have agreed upon that form for one of the others, so we must not use it for addition. Try again. (After the scholars have thought a little.) Have you ever in arithmetic expressed the addition of numbers? (Several hands.) How?

P. By the sign plus.

T. Show us on the board with the numbers 9 and 6. (Pupil writes on the board $9 + 6$.) What does that suggest about adding *a* and *b*? (Pupils write $a + b$.) That is right. Did you think it was going to be so easy? Now, what I want you particularly to bear in mind is this: in algebra adding in this way, by the sign +, is addition just as much as adding by coefficients is; it is just as good addition to put *a* and *b* together in the way you have done, as it is to add *a* and *a* together and call it $2a$. In the latter case it is written a little more compactly, and that is all. I have told you before that it is the usage in algebra to write every quantity in the most condensed way possible, and yet have it clear; and that is the reason why we write $2a$ instead of $a + a$, which means the same thing. We cannot so condense *a* and *b*, so we write their sum $a + b$, and it should be regarded as a single quantity just as much as $2a$ or $3a$.

NOTE. This point has been emphasized, both because of its importance, and because it is so often taught wrong. I have read

in books, and heard teachers taking great pains to explain, that "a and b cannot be added, and so their sum can only be expressed, while a and a can be added." This is a distinction without a difference, and is a misconception that leads to no end of difficulty.

T. John had a marbles and lost b of them; how many had he left? P. $a - b$.

T. Yes; and what I have said of addition applies with equal force to subtraction: $a - b$ must be regarded as a simple quantity, the result of subtracting b from a. What would be the numerical result if a equals 8, and b equals 5?

P. 3.

(The pupils now rapidly do the following.)

DRILL 1.

T. Add a, b, and $2\,d$ together.
Add $3\,a$, $4\,b$, and $5\,d$ to their sum.
Subtract $4\,a$, $2\,b$, and $3\,d$ from that.
What is left? Value if $b = 5$, and $d = 6$?

Willie had b apples, and he bought $a + 3\,b$ more; how many did he then have?

Another boy joined him who had $3\,a + d$ apples; how many did both have?

They sold $4\,a + 4\,b$ of them; how many did they have left.

NOTE. See that the terms are properly arranged; when possible, putting them in the order in which the letters occur in the alphabet. Some of the pupils may have the answer to the last written thus:

$$\begin{array}{ccccc} 4\,a & + & 4\,b & + & d \\ 4\,a & + & 4\,b & & \\ \hline 0 & + & 0 & + & d \end{array}$$

T. When in subtracting one quantity from another, you subtract the whole of a term and have nothing left, do you think you need to write the zero? Is that in accord with what I have said about avoiding every unnecessary mark? (Pupils ask what "term" means.) When a quantity is made up of parts connected by the signs plus or minus, we sometimes want to refer to one of the parts, and we call it a term, as the term $4a$, or $4b$, in the quantity $4a + 4b + d$. It is a convenient word, and you will become acquainted with it by use. (Teacher repeats question.)

P. (After a little thought.) No, sir.

T. In what other way than by writing the cipher could you indicate that $4a$ from $4a$ left nothing? (Pupils think.) What would it show if you didn't write down anything whatever?

P. That there was nothing left.

T. Wouldn't that save something? P. Yes, sir.

T. Then, do this in that way. From $5a + 6d + 7c$ subtract $3a + 6d + 5c$. Write the answer as compactly as possible.

Add 6 to $3a$. (Some of the pupils will probably get the answer $9a$. If so, the teacher will continue.) I see that some of you have got the answer $9a$. Now let us see what they have done. How much must be added to $3a$ to get $9a$? . . . Why, that is simple. How many must be added to 5 to make 8?

P. 3.

T. How many must be added to $2x$ to produce $5x$?

P. $3x$.

T. Well, then, what must be added to $3a$ to get $9a$?

P. $6a$.

T. But I asked you to add only 6, not 6 a. What will the sum be?

P. $3a+6$.

T. Certainly. Subtract 5 from m. P. $m-5$.

T. Subtract d from 7.

From $4a+9c+8$, subtract $4a+9c+6$.

From $7+6a+d$, subtract $7+3a+d$.

Add 7 a, 3 b, 5, 6 a, 7, and 4 b, together.

Divide 12 c into two such parts that one of them is 5 times the other.

Note. If the pupils cannot readily do this, refer back to some of the problems in the preceding lesson.

What is the sum of m and n? What is the difference?

Mr. A has a farm of c acres, and Mr. B has one of d acres; how much larger is Mr. A's farm than Mr. B's?

Find the numerical value of x in the following equations, if $a=5$ and $b=4$.

$$x=4a+5b-20$$

$$x=40-6b+a.$$

T. Make me a quantity six times as large as x. (The pupils all write $6x$.) Five times as large as a. (It is written by all.) Seven times as large as y. Eight times as large as x. Six larger than x. (The pupils all write $6x$.) What is your answer?

P. $6x$.

T. Well, that is queer. What quantity is six times as large as x?

P. $6x$.

T. Is a quantity 6 larger than x, and 6 times x just the same? would 6 times 7 be just the same as a quantity 6 larger than 7. (Pupils see the dilemma.) Find a quantity 5 larger than 12. (To all these the pupils give answers.) 8 larger than 20; 9 larger than 15; 6 larger than x (if the pupils hesitate, the teacher continues); 6 larger than 7; 6 larger than 13; 6 larger than 15; 6 larger than x (and so on till the pupils get $x + 6$). Make me a quantity 7 times as large as x; 8 larger than x; 5 larger than a; 5 times a; 7 larger than b; 7 times b; 7 smaller than b (the pupils hesitate or blunder); 7 smaller than 20; 9 smaller than 15; 7 smaller than b (the pupils get it $b - 7$); 8 smaller than x, etc.[1]

In beginning a new study work slowly, and give the pupils plenty of time to think. The new ideas have to break out, as it were, a path for themselves through the ganglia of the brain, and the better the path in the beginning, the greater the speed in the end.

LESSON III.

Multiplication and Division of Positive Quantities — Exponents.

TEACHER. Add 5 a, 7 a, and 8 a. Subtract from the sum 12 a.

Add m and n. Subtract m from n. Subtract n from m.

Add a and a; subtract a from a.

Multiply a by a. . . Why don't you write the answer?

PUPILS. We don't know how.

T. Well, that's reason enough. Now, scholars, as I have told you before, if we are to work out mathematical problems by means of letters, we must be able to add, subtract, multiply, and divide with them. Of the first two we have learned something; now, how shall we express multiplication, as, for example, a times a? Who can suggest a way? (Several hands.) Show us.

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[1] Etc. means continue drill till pupils acquire proficiency.

P. (Writing on the board.) $a \times a$.

T. That certainly does indicate that a is multiplied by a. But you will remember what I have said about finding the simplest and shortest possible way to write or express results. Now, I will admit that $a \times a$ indicates what we want, but is it the simplest way? Have we been content to write the sum of a and a this way, $a + a$?

P. No, sir.

T. How have we condensed it?

P. Into $2\,a$.

T. What, then, does $2\,a$ mean?

P. Two a's added together.

T. And $3\,a$?

P. Three a's added together.

T. And so on. Now, can't we in some way condense the expression for multiplication that John has given us? (A hand.) Show us.

P. $2\,a$.

T. Well, I cannot say I think that would be well. We use that to express the sum of two a's; and if we also use it to express the product of two a's, how shall we know which is which? Or, perhaps pupil thinks they are always the same. Let us see. What is the sum of 2 twos? (The pupils answer promptly this and the following.). . . The product of 2 twos? . . They are certainly the same in this case. . . . The sum of 2 threes? . . . The product of 2 threes? . . . Here they differ. . . . The sum of 2 fours? . . . The product of 2 fours? . . . The sum of 2 fives? . . . The product of 2 fives? . . . Well, with one exception, they are not the same; now, if we write them alike, will you tell us how we are to know which is meant?

P. I don't think they should be written alike.

T. Neither do I. Can any one suggest a better, or another way? (Some hands.)

P. Put it after the letter. (Writes on the board *a* 2.)

T. That's pretty good, but still there is an objection to it. We sometimes, as you will soon see, have to write two or more letters together in this way; *abc*. Now, if some one should write *a* 2 *b*, how could we know whether it was two *a*'s multiplied together, or two *b*'s added together?

P. Put a dot after the 2 in this way, *a* 2.*b*.

T. Well, that might do; but without detaining you any longer, I will show you how it is expressed, and I think you will admit that it is an improvement. Put it after the letter as John suggests, and also a little above, thus (writing a^2). You must, I think, thoroughly understand by this time that this expression means two *a*'s multiplied together; what does this (a^3) mean?

P. Three *a*'s multiplied together.

T. And this (a^5)?

P. Five *a*'s multiplied together.

T. I want to multiply this (a^3) by *a*. That means that I have three *a*'s multiplied together, and I want to multiply in another, what will be the result?

P. Four *a*'s multiplied together.

T. Write it. (Pupils do so, a^4.) Before going any farther, we had better learn to read these quantities. First let us write out these expressions (a, a^2, a^3, a^4, a^5, a^6, a^7, etc.). This figure (or quantity of any kind), written as you see here, to show how many times another quantity is taken as a factor, is called an *exponent* of such a quantity (writing the word on the board). ^{7}, ^{6}, ^{5}, ^{4}, etc., are here exponents of a. The quantities that are produced in this way are called powers of the quantity by which they are produced. For example, this (pointing to a^7) is the seventh power of a, which is only another and very convenient way of saying that it is formed by taking a as a factor 7 times. What is this, scholars (a^6)?

P. The sixth power of a.

T. And this (a^5)?

P. The fifth power of a.

T. And this (m^9)?

P. The ninth power of m.

T. What does it mean?

P. That m is taken as a factor nine times.

T. Write it out in the way that John suggested for expressing multiplication. (All write it out as follows: $m \times m \times m \times m \times m \times m \times m \times m \times m$.) Now, which do you think the simpler, to write it in this way, or to express it by an exponent? (Pupils all agree that to express it by an exponent is the better way.)

T. There is one term here where there seems to be no exponent (pointing to a). How do you account for it? (The pupils hesitate). How many times is a a factor in that quantity?

P. Once. (All are made to see this.)

T. Then what should be the exponent?

P. 1.

T. Do you see why it can be omitted?

P. Because it isn't necessary to write it when it is 1. You can tell that by a itself.

T. Right. Some more algebraic economy, you see; never to waste any time, pencil, or paper. But now, scholars, let me tell you how to read them. I have told you that this (a^7) means the seventh power of a, but that is too long an expression, so in reading it is customary to say simply a seventh; this one (a^6) a sixth. Read this one (a^5).

P. a fifth.

T. And this one (a^4)?

P. a fourth.

T. And this one (a^3)?

P. a third.

T. This one (a^2), is of course a second, but the more common usage is to call it a square. This one (a^3), is often called a cube. This one (a) is called simply a. Read these quantities (this the pupils do as the teacher writes them):

b^6, c^5, d^4, c^2, d, d^3, m^{10}, n^8, m, m^2, m^3, x^2, x^3, etc.

T. Multiply a^3 by a^2; now, scholars, notice this: a^3 means three a's multiplied together, and a^2 means two a's multiplied together; if we multiply them together, how many will there be in all?

P. Five a's.

T. Then write it. (Pupils do so and read result.)

DRILL 2.[1]

Multiply m^2 by m^{10}; c^4 by c^8; g^5 by g; a by a^4, etc.

T. How have you been able to tell in all these cases what the exponent of the product is?

P. By adding.

T. There you are talking piecemeal again. By adding what?

P. By adding the exponents.

T. Well, that is just as vague. The exponents of what?

P. The exponents of the quantities you want to multiply together.

T. "Quantities you want to multiply together," seems to me very awkward. Can't you express the idea better? What are the names of quantities to be multiplied together? Class.

P. Multiplicand and multiplier.

T. See if you cannot simplify your statement. (Repeats question.)

P. By adding together the exponents of the multiplier and multiplicand.

T. That is better, but I would prefer to say by adding together the exponents of the factors. For example, multiply together c^2, c^5, and c^3; d^4, d^3, and d^2, etc.

T. Once more, a^5 means that a is taken as a factor 5 times. Now, if I divide it by a it would take out one of them, wouldn't it? P. Yes, sir.

T. Then what would be the quotient, that is, how many a's would be left?

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[1] In these drills the pupils, of course, give the answers.

P. Four a's.

T. Write it. (They write a^4.) Now, if I had divided a^5 by a^2, how many of the a's would it take out?

P. Two of them.

T. How many a's multiplied together would there be left?

P. Three a's. (They are made to write it, a^3.)

DRILL 3.

T. Give me quickly answers to the following:

Multiply b^3 by b^5.
Divide the product by b^4.
Divide c^7 by c^5. Multiply the quotient by c^3.
Divide m^8 by m^2.
Divide m^{10} by m^4.
Divide m^8 by m^7.
Divide d^4 by d^3.

T. How have you, in these cases, found the exponent of the quotient? Now, all think before you speak and say the first time what you mean. You know the names of the parts in division, do you not?

P. Yes, sir.

T. What are they?

P. Divisor and dividend.

T. Well, look at your work and tell me how you have found what the exponent of the quotient is. (After a little thought hands are raised.)

P. I subtract the exponent of the divisor from the exponent of the dividend.

T. Just the opposite of multiplication, you see. Now for some quick work.

QUESTIONS.	ANSWERS BY PUPILS.
x^4 multiplied by x^3 ?	x^7.
x^4 divided by x^3 ?	x.
a^5 divided by a^3 ?	a^2.
a^5 divided by a^4 ?	a.
a^5 divided by a^5 ? (Pupils hesitating.)	1 ?

T. Yes; but why didn't you subtract exponents? That would give you a^0. (Pupils look puzzled.) Well, it would be all the same, a^0 or any letter with zero for an exponent means one; and I could call 1, a^0, b^0, x^0, or any other letter with zero for its exponent, if I wanted to.

NOTE. After a little more drill in multiplying and dividing, the lesson closes with the following exercise, the pupils writing the quantities that the teacher calls for.

Write a number 7 larger than y; 9 smaller than x; 17 larger than x; 25 larger than a; 15 smaller than x; John has x cents and Henry has 5 more; how many has Henry? (Pupils write $x + 5$.) How many have both? (Pupils write $2x + 5$.) If we know that that equals 35 cents, what equation can we form? (Pupils write the equation $2x + 5 = 35$, after a little thinking.) Now, scholars, tell me, if you had the equation $3x = 24$, how you would get rid of the 3 by which x is multiplied?

P. Divide by 3.

T. Very well; 3 in 90, 30 times,
3 in 93, 31 times,
3 in 96, 32 times.

How long shall I have to keep on?

P. I meant divide both members of the equation by three.

T. Then, as I have already hinted a number of times to the class, that is what you should have said. So, if a quantity is connected with x by multiplication, to what process do we resort to get rid of it, or, as we shall say, *eliminate* it?

P. Division.

T. Then, if a quantity is connected with x by addition, how can we eliminate it? (Pupils hesitate.) When a quantity is *multiplied* into x, we divide it out of the way; what ought we to do, when it is *added* to x, to eliminate it?

P. (Inquiringly.) Subtract it?

T. That would be according to reason, would it not?

P. Yes, sir.

T. When I eliminated a quantity multiplied into x, I divided both members; what shall I do to eliminate a quantity by subtraction?

P. It should be subtracted from both members.

T. Right; then let us return to the equation $2x + 5 = 35$, and solve it. How shall I eliminate the 5?

P. Subtract it from both members

T. What does it give?

P. (Performing the work.) $2x = 30$.

T. Now eliminate the 2 by division.

P. $x = 15$.

T. Who had the 15 cents?

P. John.

T. How many had Henry?

P. $x + 5$, or 20.

T. William has a number of postage stamps, and Mary has 50 less; together they have 2550; how many has each? First tell me how many William has.

P. x.

T. How many has Mary if she has 50 less?

P. $x - 50$.

T. How many have both?

P. (Carefully.) $2x - 50$.

T. What does that equal? Make the equation.

P. (Writing.) $2x - 50 = 2550$.

T. How shall we eliminate the 50 that is connected with x by subtraction? P. By addition.

Note. Finish the example. In solving an equation of this kind, $2x - 50 = 2550$, *do not* "transpose," thus: $2x = 2550 + 50$. Perform the operation when possible at once, as: $2x = 2600$. The other is not scientific. You might as well teach the roundabout course when division is used, thus: —

$$\begin{aligned} 2x &= 12 \\ x &= 12 \div 2 \\ x &= 6. \end{aligned}$$

Do not use the word transpose in this process. Or, if you do, simply explain that it is a word the pupils may find in some algebras. It should be discarded, as it misleads the scholars unless a good deal of explanation is given and repeated frequently to show that transpose does not mean transpose: —

"That I, Pyramus, am not Pyramus."

Nothing is more absurd than the way those who use the word transpose take to justify its use. For example, take $x + 5 = 25$. The pupils have studied arithmetic, and presumably know as well how to subtract 5 from 25 as from $x + 5$; but, nevertheless, the 5 must be transposed, and the work must be done thus: $x = 25 - 5$.

When the equations contain several terms to be eliminated from both members, it may be convenient to add and subtract some or all of them by signs, but not when it can be avoided. As, in raising $3a + 2b$ to the 4th power, it may be well to express it at first thus: $(3a)^4 + 4(3a)^3(2b)$, etc., but no one would think of doing this with $x + y$, or whenever it could be conveniently avoided. (See Preface.)

The simpler the illustrations, the more readily the student grasps the principle.

LESSON IV.

Multiplication and Division continued — Fractional Expressions.

TEACHER (calling up the pupils individually to answer the questions, which should be put rapidly and answered promptly).

Multiply a^3 by a^2; divide a^3 by a^2.
Multiply x^5 by x^3; divide x^5 by x^3.
Multiply $6a^3$ by $2a^2$. (The pupils hesitate.) Do the coefficients of a^3 and a^2 bother you?

PUPILS. Yes, sir.

T. What would $6a^3$ multiplied by 2 give you?

P. $12a^3$. (Some of them see the way to do the original example, but the teacher, to make sure, keeps on.)

T. What will $12a^3$ multiplied by a^2 give you?

P. $12a^5$.

T. How many see how the answer was obtained? (All raise their hands.) Then, to show it, multiply $5x$ by $3x$. (Pupils quickly give the answer.) Then divide $6a^3$ by $2a^2$. (After a moment's study all the pupils get the answer, $3a$.) All, then, rapidly: —

Multiply $9a^4$ by $3a^2$. Divide $9a^4$ by $3a^2$.
Multiply $12a^5$ by $4a^5$. Divide $12a^5$ by $4a^5$.

Multiply $7\,x^3$ by $7\,x^3$. Divide $7\,x^3$ by $7\,x^3$.
Multiply a by b. (Pupils hesitate again.)

T. You evidently see that the product of a and b cannot be either a^2 or b^2. Yet, if we are to use letters in mathematical problems, we must have a way to multiply different letters, as well as the various powers of the same letter. Now, as we cannot write a and b into one letter, I would like to have you show your ingenuity by thinking out a way to express the product of a and b. (Several hands go up and a number of the class suggest $a \times b$.) That certainly means a multiplied by b. The only question now is, can it be simplified, or written more compactly? (Provided some pupil, after a little thinking, does not suggest it, the teacher shows that the sign between the letters can be omitted, and that ab indicates the product of a and b, and tells the pupils how to read it.) Now, scholars, write the product of: —

a multiplied by c; $2\,a$ multiplied by d; $3\,x$ multiplied by a (show the pupil how to arrange the letters); $2\,ab$ multiplied by a^2 (see that they write this correctly).

a^2x multiplied by a^3x^2; $5\,a^3b^2$ multiplied by a^2c;
$3\,x^3y$ multiplied by $2\,a^2x^2y^2$; $7\,ab^4$ multiplied by $3\,a^3x^2$;
$5\,x^2y$ multiplied by $4\,x^3y^2$; $8\,mn$ multiplied by $4\,m^2x$.

Note. Continue this drill till all get the process.

T. Now, scholars, divide: —

a^4 by a^2; a^3b^2 by ab^2; a by a;
a^3b by a^3y; x by x; b^2 by b^2;
$6\,x^5y^2$ by $3\,x^3y$; $9\,a^7b^7$ by $3\,a^5b^5$; a by b.

(The scholars are again brought to a stand, but after a little urging some of them write $a \div b$.) Yes, that expresses it, but again we must ask, can the form be simplified? For example, in arithmetic, how have you indicated, say, 2 divided by 3? (The pupils write $2 \div 3$, and $\frac{2}{3}$.) Which is the more convenient of these ways? the one you used the most? (All say $\frac{2}{3}$.) Then write a divided by b that way, and you will have the algebraic form of division. (All write $\frac{a}{b}$.) Now write down the answers to the following:

x divided by 6; 6 divided by x;
a^2 divided by bc; bc divided by a^2;
m^3 divided by n^2; m^2 divided by n^3;
$3x^2$ divided by $3ac$.

(The pupils write this $\frac{3x^2}{3ac}$.) Scholars, if you wanted to write the quotient found by dividing 8 by 12, would you leave it this way, ($\frac{8}{12}$)?

P. No, sir; we would write it $\frac{2}{3}$.

T. Then, will you leave the division of the last example I gave you $\frac{3x^2}{3ac}$? (The pupils all change it to $\frac{x^2}{ac}$.) How would you change this expression, $\frac{ab^2}{b^3}$? (All suggest $\frac{a}{b}$ after a little study); and this one, $\frac{3ax}{6ab}$? (All change it to $\frac{x}{2b}$.) Just like the process you have already learned in arithmetic, is it not? P. Yes, sir.

Note. A general response may be well for a few of the first of the following examples, but, as a rule, it is better to call for individual replies, heeding a suggestion already given several times.

T. Find the answers to these examples: —

a^2bc multiplied by $2\,ab^2c$; a^2bc divided by $2\,ab^2c$;
$6\,x^2y$ multiplied by $2\,xy$; $6\,x^2y$ divided by $2\,xy$;
$12\,m^3$ multiplied by $4\,m^2$; $12\,m^3$ divided by $4\,m^2$;
$16\,abc$ multiplied by $32\,a^2b^2c^2$; $16\,abc$ divided by $32\,a^2b^2c^2$.

Multiply $a^2 + 2\,ab + b^2$ by a^2c^2; divide product by ac^2.

Multiply $2\,ab + 4\,a^2b + 8\,ab^2$ by a^2b^2; divide product by a^3b^3.

Divide $3\,ab + 6\,a^2b^2$ by $3\,a^2b^2$.

Divide $4\,x^3 + 8\,x^2 + 2\,x$ by $4\,x^2$.

Let the pupils learn the technical terms by using them. In this way we learn all the words we have the best command of. Use them first in connection with some explanatory phrase, as "eliminate" in Lesson III. Do not bother your pupils with set definitions.

LESSON V.

Theory of Negative Quantities.

TEACHER. Three men went fishing. A caught five times, and B three times, as many fish as C; together they caught $18\,a + 27\,b$ fish, how many did each catch? (After a sufficient time is allowed, the teacher calls on them individually till he gets the following result.)

PUPIL. A catches $10\,a + 15\,b$; B catches $6\,a + 9\,b$; and C catches $2\,a + 3\,b$.

T. If a is 3, and b is 1, how many would each have? (The pupils compute the number.)

TEACHER.	PUPILS.
$7a$ from $12a$?	$5a$.
$3a$ from $21a$?	$18a$.
$6b$ from $4a$?	$4a - 6b$.
$7b$ from $3b$?	

(After studying a little on this example, some of the pupils raise their hands.)

T. Well, what is the matter?

P. You can't do it.

T. I can't do what?

P. You can't take $7b$ from $3b$.

T. Do you mean that *I* can't, or that *you* can't?

P. (Some of them.) We can't. (Others.) Nobody can, because $7b$ is greater than $3b$.

T. Well, let us see. But before we settle this question, there is something that you must learn. (Writing 5 on the board.) What is that?

P. Five.

T. Give me the number one less.[1]

P. Four.

T. (Writing the 4 to the left of the 5.) Give me the number one less than four.

P. Three.

T. (Writing 3 as before.) Give me the number one less than three. P. Two.

T. (Writing it.) Give me the number one less than two. P. One.

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[1] The terms "greater" or "less" as here used mean positively greater or less.

T. (Writing it.) And one less than one.

P. Zero.

T. (Writing it.) And one less than zero.

P. There isn't any.

T. Well, let us see.

Note. The series as now written on the board is: 0, 1, 2, 3, 4, 5.

T. You have already learned in algebra some very strange things, and it isn't safe to say what can or cannot be, without a good deal of thinking. I know, too, that there are some wise men who say, as you do, that there are no quantities less than zero; but wise men sometimes err. Tell me which is the smaller, seven or five?

P. (Unanimously.) Five.

T. How much smaller?

P. Two smaller.

T. Very well; now tell me what seven is. What does it mean? P. Seven units.

T. Seven units! How much do they weigh? How long are they? How much space do they occupy? (The pupils begin to think. The teacher continues to press for an answer.) Come, tell me. I will give you seven, twenty, a thousand, if you wish. You see I am very generous. Can you carry them home, or shall you need an express-wagon? What do you think? (Some of the pupils raise hands.)

P. We can't say, unless you tell us what it is a thousand of.

T. I do not mean a thousand of anything. I simply mean a thousand. When I asked you which is the smaller, seven or five, I didn't say seven or five of anything, and yet you answered very confidently that five was two smaller. I mean these numbers as abstract, not concrete. You know what abstract numbers are, do you not? (If the pupils have not learned this distinction it should now be made clear to them.) Now, what I want to know is this: What is the actual value of the abstract number five? (After a little discussion it is decided that it is nothing.) And what is the actual value of the abstract number seven?

P. Nothing.

T. Then, according to your statement, you have a "nothing" that is two smaller than another "nothing." (The pupils see that by speaking thoughtlessly they have been caught, and they have to admit it.) My purpose, scholars, in this discussion was twofold. First, you were so confident that there could be nothing smaller than zero, that I wanted to shake your ideas up a little, and show you that possibly there are some things about numbers that you have not thought of. Second, and chiefly, I wanted to impress upon you that the actual value of numbers depends on the concrete things they enumerate. We have learned that the actual or absolute value of all abstract numbers is zero. There is one thing more to which I want to call your attention; it is that the absolute value of a number, when applied to different concrete objects, is not always the same, but depends on the object. I would rather have *one* bushel of anything worth having than *one* quart, or even seven quarts, wouldn't you?

P. Yes, sir.

T. The *number* of the things may be greater, and in arithmetic it is the number that is emphasized; but the *absolute value* depends on the concrete things enumerated, and in algebra it is the absolute value that is emphasized. If in arithmetic I speak of ten marbles, I am thinking of the number, but in algebra, if I say b marbles or $5\,b$ marbles, I am thinking of it as a quantity of marbles without reference to the number. Remember, also, what I have already told you; that the absolute or actual value of a quantity depends on the thing named, or, as we may say, the surname of the quantity.

Now let us approach our object from another point.

The first day I deposit four dollars in a bank; the second, three dollars; the third, two dollars; the fourth, one dollar; the fifth, no dollars; now, scholars, could I do less than that on the sixth day? (The pupils think.) You see, do you not, that each day the money I had in the bank was one dollar less than it would have been if I had put in the amount of the day before? That is, when I put in no dollars, my money in the bank was one dollar less than if I had put in one dollar, as the day before. Do you see that?

P. Yes, sir.

T. Now, could I do anything the sixth day that would make my money in bank one dollar less than if I had merely deposited no dollars?

P. (Readily). You could take out one dollar.

T. Yes; and in arithmetic what would it be called?

P. (After some discussion.) A withdrawal.

T. Yes; to illustrate: in arithmetic we would speak of a deposit of four dollars, of three dollars, of two dollars, of one dollar, of nothing. This would mean putting in four dollars more than nothing, three dollars more than nothing, two dollars more than nothing, one dollar more than nothing, and, finally, nothing. If we kept on, we would speak of a withdrawal of one dollar, meaning a withdrawal of one dollar more than nothing; of a withdrawal of two dollars, *i.e.*, of two dollars more than nothing, etc.

What I want you to notice, scholars, is this: that there are two transactions having opposite effects on a bank account, called in arithmetic by different names; one, a deposit, increases the account, and the other, a withdrawal, decreases it. They meet and pass into each other at zero.

Now, the peculiarity of algebra is that we can treat both of these transactions under the same name. We can call them both deposits, or both withdrawals, just as we choose. The question now is, what is the value of a withdrawal, as compared with zero, if we call it or consider it a deposit? I will grant that a withdrawal of one dollar means that one dollar more than nothing has been taken out; but if it is called a *deposit*, then how much has been put in? A good way to find out is to consider the effect on a bank account.

Suppose I have been adding a dollar a week to my bank account, till finally I skip a week; how much smaller will my bank account be than if I had kept on?

P. One dollar less. T. Why?

P. Because nothing is one dollar less than one dollar.

T. Tell me how much I decreased my deposit in these cases: —

I have been putting in $5.00 per week, and at the end of a certain week my account stands $200.00. At the end of the next week it is $204.00. How much less than usual did I put in that week? P. One dollar less.

T. How do you know that?

P. Because your bank account is one dollar less than it would have been if you had put in $5.00.

T. I have been depositing $4.00 per week, till my bank account is $40.00. At the end of the next week it is $43.00. How much less than usual did I put in?

P. One dollar less.

T. How do you know that?

P. Because your account is one dollar less than it would have been if you had put in $4.00.

T. I have been putting in nothing per week, and my bank account is $30.00. At the end of the next week it is $29.00. How much less than usual did I put in, or deposit?

P. You took out one.

T. You've gone back to arithmetic. I do not want to know how much I took out; I want to know how much *less* than my usual deposit I *put in?*

P. One dollar less.

T. How do you know that?

P. Because your account is one dollar less than it would have been if you had put in nothing.

T. But my usual deposit was nothing, so you see there is such a thing as a *deposit less than zero.* How large a deposit as compared with zero would a withdrawal of two dollars be?

P. A deposit of two dollars less than zero.

T. A withdrawal of three dollars?

P. A deposit of three dollars less than zero. (Continue this up to a deposit of ten dollars less than zero.)

T. But, scholars, we can call these transactions all withdrawals if we chose; then the withdrawals are greater than nothing; but what will the deposits be if we call them withdrawals? (The pupils hesitate.) That is, we may speak of a withdrawal of $3.00, $2.00, $1.00, or nothing, meaning that we took out three dollars, two dollars, one dollar, or no dollars, more than nothing; but if we put in a dollar and called it a withdrawal, would it be taking out a dollar more than nothing, or a dollar less than nothing?

P. A dollar less than nothing.

T. How much of a withdrawal will putting in two dollars be?

P. Two dollars less than no withdrawal.

T. I will name some quantities, and I want you to give me their value in the opposite name. Four dollars deposit.

P. A withdrawal of four dollars less than no dollars.

T. Five dollars withdrawal.

P. A deposit of five dollars less than no dollars.

T. A withdrawal of seven dollars less than nothing.

P. (After a little stumbling.) A deposit of seven dollars.

T. A deposit of six dollars less than nothing.

P. A withdrawal of six dollars.

Note. Continue this drill till the pupils are quick in responding. Then put on the board some marks thus, | | | | |, fifteen or twenty of them.

T. I can do two opposite things to these marks, add to them, thus, | |, or subtract from them, thus (erasing two or three). Now, in algebra, I can call both processes adding or subtracting, just as I choose. Let us see how it works: I want to add three; tell me how.

P. Make three more marks.

T. Yes (doing so). Now, I want to add four less than no marks; how can I do it?

P. (After a little hesitancy.) Erase four.

T. (After erasing the four.) I want to subtract five.

P. Erase five.

T. (Doing so.) I want to subtract six less than no marks.

P. (Eagerly.) Add six.

Note. Send the pupils to the board, each making a certain number of marks, and give them exercises of this kind till they can perform them readily. While a part, say a third, of the class work at the board, the rest can watch for mistakes.

T. Quantities less than zero, scholars, are called negative quantities, and those greater than zero positive. The question now arises, how shall we write them and read them to show that they are negative quantities, or less than zero? Let us return to the series with which we began this discussion (0, 1, 2, 3, 4, 5), and continue it as far below zero as it commences above. How shall we write in this series the quantity one less? (Slowly a few hands come up.)

P. Write it before the zero.

T. There is where I want it written. Show us how. (Pupil goes to the board and writes thus, 1, making the series stand 1, 0, 1, etc.) Yes; but how shall I know that it is 1 less than zero? I may want to use it in a problem by itself as I do other numbers. How shall I distinguish it from 1 greater than zero? As they stand now, they are written just alike.

NOTE. The pupils see the dilemma; and, after piquing their curiosity a little longer, and drawing from them a few suggestions, the teacher explains that it is done by writing this sign, $-$, before a quantity and reading it minus, as: -1, minus 1; $-a$, minus a, etc.

T. Read these quantities (writing on the board such expressions as b, $-b$, $-2c$, -8, etc., and having the pupils read them and tell how much greater or less than zero they were. This should be continued till the pupils become familiar with the expressions).

T. Now, scholars, I want to introduce some other illustrations of positive and negative quantities, mostly to clear up your ideas regarding them, but partly to correct some false teaching that you may run against. It is quite common to hear teachers say, for example, that north latitude is positive and south latitude negative; that property or assets is positive, and debts negative. This is wrong. South latitude, when counted south, is not less than zero, any more than north latitude is; and debts, when treated as debts, are greater than zero, just as much as assets are. Let us now search for the true explanation in these cases.

NOTE. Use a wall map of one of the hemispheres, or, in want of this, draw a number of parallel lines on the board, to represent the equator and degrees north and south.

T. (Pointing to the map.) What latitude is this?

P. Ten degrees north.

T. And this?

P. Five degrees north.

T. And one less?

P. Four degrees north.

T. And one less?

P. Three degrees north.

T. And one less?

P. Two degrees north.

T. And one less?

P. One degree north.

T. And one less?

P. No degrees north, or on the equator.

T. And one less?

P. One degree south.

T. Yes; that means one degree more than zero degrees south, and in arithmetic we would so state it. But I don't want to think of it as south; I want to think of it as north, and in algebra we can if we wish. Now, the point to be settled is this; if we call it north, how does it compare in distance north with the equator? Is is farther north or not? In other words, is it more or less than zero degrees north?

P. Less than zero degrees north.

T. How much less?

P. One degree less.

T. How can we write it?

P. (Going to the board, writes and reads) – 1° north.

T. (Pointing to southern parallels.) How far *north* is this?

P. Minus five degrees north.

T. And this?

P. Minus ten degrees north.

NOTE. After some more drill like this, show that all latitude can be expressed as south, in which case north latitude is minus so many degrees south. Perhaps the best illustration of this principle is by assets and debts; and the teacher should develop this illustration, showing that debts may be expressed as assets less than zero, and assets may be expressed as debts less than zero. The thermometer, used by some teachers, is not so good to begin with, for the reason that while we sometimes write the temperature algebraically, we always read it arithmetically; as, for example, we say ten degrees above zero, ten degrees below zero. Algebraically we should read, ten degrees above zero, *minus* ten degrees above zero, etc.

(The teacher should now call attention to the fact that concrete algebraic series are in pairs, the reverse of each other, and that the positives of one have the same absolute value as the negatives of the other having the same number of units. For example, plus four dollars of assets is the same as minus four dollars of debts.[1])

T. To the series of quantities already on the board, we will now add two others. First, under 5 I will write $5a$; what quantity is a less than $5a$?

P. $4a$.

---

[1] The absurd statement found in some algebras that $+3$ and -3 have the same absolute values, is probably a confusion of this fact. If $+3$ and -3 have the same absolute value, then a man three dollars in debt has the same absolute sum of money as one having three dollars, or a ship in three degrees north latitude would be in the same latitude as one minus three degrees north. Three dollars of debt has the same absolute value as minus three dollars of assets, and three degrees north is the same as minus three degrees south.

T. (Sending a pupil to the board.) Write it under 4. What quantity is a less than $4a$?

P. $3a$.

T. Write it under 3. (This is done, and in the same way all the following quantities are written in proper order.) What is the quantity a less than $3a$?

P. $2a$.

T. Write it. What is the quantity a less than $2a$?

P. a.

T. And a less than a? P. Zero.

T. And a less than zero? P. $-a$.

T. And a smaller still? P. $-2a$.

Note. Continue this to $-5a$.

T. Under $-5a$ I will write $-5b$; who can write after it a quantity b larger? (Hands go up, and a pupil is called who writes $-4b$. In this way a series of b's is completed.)

Note. The series on the board will now be as follows: —

$$-5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$
$$-5a,\ -4a,\ -3a,\ -2a,\ -a,\ 0,\ a,\ 2a,\ 3a,\ 4a,\ 5a.$$
$$-5b,\ -4b,\ -3b,\ -2b,\ -b,\ 0,\ b,\ 2b,\ 3b,\ 4b,\ 5b.$$

These series may be extended farther, both up and down, if the teacher thinks best, and eventually it will be well to do so.

Suggestion. In the following drill the counting process will sometimes be of assistance. For example, from $-3a$ to $4a$ counts $7a$; hence, $4a$ is $7a$ larger than $-3a$. Counting downward shows how much smaller one quantity is than another.

DRILL 4.

T. (Pointing out the quantities.) Which is the larger, 4 or 1? P. 4.

T. How much larger?
P. 3 larger.

T. Which is the larger, 3 or 0?
P. 3.

T. How much larger?
P. 3 larger.

T. Which is the larger, 3 or -1?
P. 3.

T. How much?
P. (Perhaps hesitating a little.) 4 larger.

T. Which is the larger, 0 or -4?
P. 0.

T. How much?
P. 4 larger.

T. Which is the larger, -2 or -5?
P. (Stumbling a little.) -2.

T. How much?
P. 3 larger.

T. Which is the larger, $5a$ or $-a$?
P. $5a$.

T. How much? (If they hesitate show them how to count it. Counting from left to right shows how much greater; from right to left, how much smaller.)

P. $6a$ larger.

T. Which is the smaller, 0 or $-4b$?
P. $-4b$.

T. How much smaller?

P. $4b$ smaller.

T. As I point out the quantities, tell me which is positively the larger or smaller as I may require, and how much. (Call the pupils individually, the slower ones first.)

DRILL 5.

TEACHER'S QUESTIONS.	PUPIL'S ANSWERS.
The larger, $-5a$ or $2a$?	$2a$; $7a$ larger.
The larger, $3b$ or $-4b$?	$3b$; $7b$ larger.
The larger, 0 or $-3b$?	0; $3b$ larger.
The smaller, $4a$ or $-5a$?	$4a$; $9a$ smaller.
The smaller, -2 or -5?	-5; 3 smaller.

Continue this till the pupils respond quickly.

NOTE. In algebra we may say any quantity is greater or less than another, as, for example, b is $-2b$ greater than $3b$; but as we have been using the terms (see footnote, p. 31), the meaning has been confined to positively greater or less. It may after a little be well to explain this to the pupils, and give them drill in it, changing the questions to "How much larger is," etc.? This should certainly be done in the high school.

LESSON VI.

Addition — Problems containing Fractions.

NOTE. Spend three or four minutes in a drill similar to that with which the last lesson closed. Then change to addition. *Send a pupil* to the board, where the work can be seen, to put down in order the following examples as they are answered, and keep them on the board through the lesson.

TEACHER. What is the sum of 3 and 4?

PUPILS. 7.

T. Of 3 and 3? P. 6.

T. Why is the sum one less than before?

P. You added one less.

T. Now you are answering in fragments again Please tell me what I added *one less than,* and to what I added it; one of the numbers is the same in both cases.

P. The first time you added 4 to 3. The second time you added one less than 4 to 3, so the sum is one less.

T. Very good and clear. If I add to 3 one less still, how will the sum compare with this? (Pointing to the 6.) P. It will be one less.

T. What is the quantity one less than 3? P. 2.

T. Add 2 to 3. (This is done and the sum compared with the preceding sum.) If we add to 3 a quantity one less than 2, how will the sum compare with 5?

P. It will be one less.

T. What is the quantity one less than 2? P. One.

T. Very well; add 1 to 3. (This is done, and the sum compared with the preceding sum.) If we add to 3 a quantity one less still, how will the sum compare with this? (Pointing to the 4.) P. It will be one less.

T. What is the number one less than 1? P. Zero.

T. Add 0 to 3. (This is done and the result compared.) If I add a quantity one less than 0 to 3, how must the sum compare with this (3)?

P. It must be one less.

T. What is the quantity one less than 0?

P. Minus one.

T. Add it to three.

NOTE. If the scholars hesitate, do not press for the answer, but review the examples which have been kept on the board, showing that if numbers decreasing by one are added to 3 (or any number), the sums decrease by one. Repeat this till the pupils see that by analogy 3 plus -1 must be 2.

T. Put it another way. If A has three dollars and B one dollar, how much money have both together?

P. Four dollars.

T. If A has three dollars and B has no money at all, how much have both? P. Three dollars.

T. If A has three dollars and B is one dollar in debt, how much would they both have if they combine and pay off B's debt. P. Two dollars.

T. But if we call the dollar that B owes a debt, it is positive; that is, it is a debt of one dollar more than nothing. What can we call it to give it a value under the new name of one dollar less than nothing?

P. We can regard it as money which he has, and say he has minus one dollar.

T. Then let us change the wording of the problem to this: A has three dollars and B has minus one dollar, how much money would both have if they united their funds? P. Two dollars.

T. (Resuming the series of examples.) If to 3 one less than -1 is added, how will the sum compare with this (2)?

P. It must be one less.

T. Give a quantity one less than -1.

P. Minus two.

T. Add it to three. . . . If to 3 one less than -2 is added, how will the sum compare with this (1)?

P. It must be one less.

T. Name the quantity one less than -2.

P. Minus three.

T. If 3 plus -2 is one, what ought 3 plus -3 to be?

P. Zero.

T. To illustrate: If you had 3 dollars and I had -3 dollars, or, as we generally say, was 3 dollars in debt, and we united our property, paying off my debt, how much money should we have?

P. None.

Drill 6.

T. Now, scholars, let me have prompt oral answers to these questions: What is the sum of 4 and 1? of 4 and 0? of 4 and -1? of 4 and -2? of 4 and -4? Of 5 and 2? of 5 and 1? of 5 and 0? of 5 and -2? of 5 and -3? of 5 and -5? of $3a$ and a? of $3a$ and 0? of $3a$ and $-a$? of $3a$ and $-2a$? of $3a$ and $-3a$? of $10a$ and $10a$? of $10a$ and $-10a$? of $10a$ and 0? of $-10a$ and 0? of $-8b$ and $-8b$? of $-8b$ and 0? of $8b$ and $-8b$? (Let the pupils now be called up individually.) Of $4b$ and $2b$? of $4b$ and b? of $4b$ and 0? of $4b$ and $-2b$? of $4b$ and $-4b$? of $4b$ and $-5b$? I see you hesitate. Well, let us again review our examples on the board. (The teacher does this slowly from 3 plus 3 to 3 plus -3, emphasizing the gradual decrease of the sum by one, as one of the components decreases by one, and then continues.) If to 3 I add one less than -3, how ought the sum to compare with zero?

P. It must be one less.

T. What is one less than -3?

P. Minus 4.

T. And one less than 0?

P. Minus one.

T. Then what must 3 and -4 together be?

P. Minus 1.

T. Add 3 and -5.

P. Minus two.

DRILL 7.

T. (Calling up individual scholars.)

Add 3 and -6
3 and -8
5 and 2
5 and -2
5 and -5
5 and -7
5 and -8
$3a$ and a
$3a$ and $-a$
$3a$ and $-3a$

Add $3a$ and $-4a$
$3a$ and $-6a$
$5a$ and $2a$
$5a$ and $-2a$
$5a$ and $-5a$
$5a$ and $-6a$
$5a$ and $-7a$
$7x$ and $-7x$
$7x$ and $-9x$
$7x$ and $-5x$

T. Now, scholars, let us return to the series of examples we have been doing on the board. We began by adding 3 to 3, and have reduced one of the numbers to -5 while keeping the other the same. Now let us continue the series, reducing the other number. 2 and -5, how many?

P. -3.

T. 1 added to -5, how many?

P. -4.

T. 0 and -5, how many?

P. -5. (The teacher may call attention to the decreasing sum as one of the components decreases.)

T. -1 and -5, how many?

P. (Perhaps hesitating a little.) -6.

T. -2 and -5? P. -7.

Drill 8.

T. (To the whole class.) What is the sum of 6 and 2? of -6 and -2? of 6 and -2? of -6 and 2? (Calling on individual pupils.)

Add $5a$ and $3a$

$-5a$ and $-3a$

$5a$ and $-3a$

$-5a$ and $3a$

$7b$ and $6b$

$-7b$ and $-6b$

$7b$ and $-6b$

$-7b$ and $6b$

$-9c$ and $-9c$

$9c$ and $-9c$

Add $5x$ and x

$-5x$ and $-x$

$-5x$ and x

$5x$ and $-x$

$7c$ and $-9c$

$4b$ and $-6b$

$-8a$ and $-3a$

$13x$ and $-13x$

$4b$ and $-10b$

$7d$ and $-8d$

T. (Using the series of numbers given in Lesson V., page 42, and calling up the pupils individually.) Add together the numbers which I point out for you. (As the teacher points out $4a$, $7a$, $-8a$, the pupils add thus: $4a$, $11a$, $3a$).

Drill 9.

TEACHER POINTS OUT.	PUPIL ADDS.
$6b$, $-3b$, $-4b$	$6b$, $3b$, $-b$
$4x$, $-8x$, $7x$	$4x$, $-4x$, $3x$
$5a$, $3a$, $-7a$, $3a$	$5a$, $8a$, a, $4a$
$6b$, $-6b$, $3b$, $4b$, $-7b$	$6b$, 0, $3b$, $7b$, 0
8, -4, -3, -5, 2	8, 4, 1, -4, -2
etc.	

NOTE. **The preceding drill will be found a good one, and a few minutes at each recitation should be given to it for several days.**

The teacher now had better put on the board a number of quantities, as $4\,a$, $-5\,a$, $7\,a$, $3\,a$, $-2\,a$, $-8\,a$, $9\,a$, in columns, and drill the scholars in rapid addition.

Do not add the positive and negative quantities separately and find their difference, but add them in the order in which they come. For example, the pupil should add the above column of quantities thus: $9\,a$, a, $-a$, $2\,a$, $9\,a$, $4\,a$, $8\,a$. Columns where the figures occupy two or more places can be added thus: First add the unit column, putting down the result; then add the tens column, putting down the result with a 0 suffixed. Then the hundreds column, putting down the result with two 0's suffixed, and so on. Lastly, combine the results.

$$\begin{array}{r} 97\,a \\ -36\,a \\ -48\,a \\ 85\,a \\ -37\,a \\ -9\,a \\ 70\,a \\ 61\,a \\ \hline \end{array}$$

Illustration of the above suggestion.

$$\begin{array}{r} 735\,x \\ -842\,x \\ 723\,x \\ -452\,x \\ 967\,x \\ -239\,x \\ \hline +2\,x \\ -10\,x \\ +900\,x \\ \hline 892\,x \end{array}$$

NOTE. **The columns of figures are rarely more than two, and pupils should be able to add two columns at once. This method will develop much greater alertness on the part of the pupils in**

observing signs and their effects than by the usual way. After a brief drill on columns of this kind, the scholars are asked to practise this by themselves, especially in doing examples in Lesson V. of the Manual.

T. Before the lesson closes I wish to call your attention to something else that I want you now to know. You have already learned that $3x$ means x multiplied by 3; $x + 3$ means 3 added to x; $x - 3$ means 3 subtracted from x; now, how shall we express x divided by 3? Who will suggest a way? (Some of the pupils may suggest this, $x \div 3$.) Yes, that means x divided by 3; but the question now is, can we not invent a simpler, or at any rate, a more compact way? In what other way have you expressed division in arithmetic?

P. (After some thinking.) By fractions.

T. Yes, I understand what you mean. This (writing $\frac{2}{3}$ on the board) means 2 divided by 3. Now, which is the simpler way to express division, this $(2 \div 3)$? or this $(\frac{2}{3})$? (All agree that $\frac{2}{3}$ is.) Then that is the best way in algebra. What does this mean $\left(\frac{x}{2}\right)$? and this $\left(\frac{x}{3}\right)$? and this $\left(\frac{2x}{3}\right)$? and this $\left(\frac{3x}{4}\right)$? Write x divided by 5, or one-fifth of x; $2x$ divided by 5, or two-fifths of x; $3x$ divided by 5, or three-fifths of x; write four-fifths of x; three-sevenths of a; five-ninths of b, etc. (till all write the quantities readily).

Now, scholars, in an equation such as $x + 4 = 7$ (writing it on the board), where I have a quantity connected with x by addition, how do we eliminate it?

P. By subtracting it from both members of the equation.

T. And in one like this (writing $x - 5 = 2$), where we have a quantity connected with x by subtraction, how do we eliminate it?

P. By adding it to both members.

T. Yes; and in one like this ($3x = 21$), where a quantity is connected with x by multiplication, how do we eliminate it?

P. By dividing both members by it.

T. Yes; and now if we have one like this $\left(\frac{x}{5} = 4\right)$, where 5 is connected with x by division, how shall we eliminate it?

P. By multiplying both members by 5.

T. Very well, what is five times $\frac{x}{5}$? (The pupils hesitate.) What is (writing the fractions on the board, the pupils answering each inquiry) five times $\frac{1}{5}$? five times $\frac{2}{5}$? five times $\frac{3}{5}$? seven times $\frac{4}{7}$? Then what is five times $\frac{x}{5}$?

P. x.

T. Five times $\frac{2x}{5}$? five times $\frac{3x}{5}$? etc.

Solve these equations:

(1) $x + \frac{x}{3} = 24$. (3) $4x + \frac{3x}{4} + 8 = 65$.

(2) $2x + \frac{3x}{5} = 39$. (4) $2x + \frac{2x}{3} - 10 = 46$.

(5) If a certain number be doubled, and four-fifths of itself and 12 be added, the sum will equal 82; what is the number?

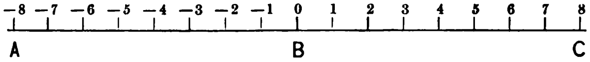
Note. If there be time left, a drill like that given on page 44 will be helpful.

In the foregoing work it may be well to learn to use the counting method of illustrating addition, using the series referred to above.

LESSON VII.

Theory of Negative Quantities continued — Addition continued — Problems.

T. Scholars, I want for a few minutes to test your understanding of the meaning of negative quantities. Some algebras that I have seen have a line drawn thus:



and we are told that distances from B to C are positive, and from B to A, negative. Now, we have taken the ground that negative quantities are quantities less than zero under the name given them. Will you tell me, if we go from B towards A as far as this first mark (− 1), if we have gone a distance less than nothing?

P. No, sir.

T. Then, if we are right in our explanation of negative quantities, can that distance from B to A be, say, —1 foot?

P. No, sir.

T. Can any of you express that distance in terms that would make it one foot less than nothing? (The pupils hesitate.) This shows me that you do not yet fully comprehend the nature of a negative quantity.

Let us return to our readings of latitude. (The teacher draws parallel lines to represent the equator and degrees north and south.) What do you call this?

P. Four degrees north.

T. And this?
P. Three degrees north.

T. And this?
P. Two degrees north.

T. And this?
P. One degree north.

T. And this?
P. No degrees north.

T. And this (pointing to 1° south)?
P. One degree south.

T. Yes; one degree more than nothing south, and so called, it is positive. But what can I call it to make it one degree less than no degrees?

P. We can call it north latitude, or degrees north.

T. Exactly. How far *north* will it take us?
P. Minus one degree north (— 1° north).

T. True. What must I call all latitude that degrees *north* may be negative?

P. South latitude.

T. Right again. Now, on this line what must I call all the distances, that those from B towards A may be negative? (Calling pupils individually till one answers correctly.)

P. We must call them all distances from B towards C. (All are made to see this.)

T. Now tell me how these distances that I point out to you compare with no distance from B towards C; as, for example, this is 6 feet more than no distance from B towards C. (As the teacher points, the pupils recite.)

P. 5 feet more than no distance from B towards C.
4 feet more than no distance from B towards C.
3 feet more than no distance from B towards C.
2 feet more than no distance from B towards C.
1 foot more than no distance from B towards C.
0 feet more than no distance from B towards C.
1 foot less than no distance from B towards C.
2 feet less than no distance from B towards C.

T. Then tell me when the distances from B towards A become negative.

P. When we call them distances from B towards C.

T. When do the distances from B towards C become negative?

P. When we speak of them as distances from B towards A.

T. Is there anything in the nature of the distances themselves that makes either of them positive or negative?

P. No, sir.

Note. The teacher now gives a brief drill, putting questions like the following: How far from B towards C is this? How far from B towards A is this? pointing out on the line various distances. The drill then changes to addition, first using the line above with such examples as this: A man starting from B went in one day 6 miles toward C and the next day 1 mile, how far did he go both days? The first day 5 and the second −2 miles; how far did he go both days? etc. Then give a drill in addition, using the series given on page 42, to which other series in *x*, or *ab*, etc.,

may now be added. At first, let the pupils respond orally as the teacher silently points out the quantities in one of the series. After a while the exercise terminates in this way: —

Drill 10.

TEACHER POINTS OUT:	PUPILS RESPOND:
$-6, 3, 5.$	$2.$
$4x, -8x, 3x$	$-x.$
$4a, \quad a$	$5a.$
$4a, \quad b$ (Pupils hesitate a moment.)	$4a + b.$
$4a, \quad -b,$ (Pupils stop and look puzzled.)	

T. Why, scholars, what is the matter? Can't any of you add $4a$ and $-b$? (No reply.) Can't some one suggest a way? (After some thinking a pupil writes $4a + -b$.) Well, that is pretty good, and certainly does mean $-b$ added to $4a$; but before we adopt that way we ought to make sure that it is the simplest way. Can't we avoid using two signs?

Let us now carefully observe how it affects a quantity to add to it a negative quantity. Of course you know that adding on a positive quantity gives a sum larger than the quantity to which it is added; as, for example, if we add 1 to 4, the sum is 5, one larger than 4; but if we add -1 to 4, what is the sum?

P. 3.

T. How does it compare with 4?

P. It is one less than 4.

T. Add $-a$ to $5a$.

P. $4a$.

T. How does the sum compare with $5a$?

P. It is a less than $5a$.

T. Then adding a negative quantity gives a sum less than the quantity to which it is added.

P. Yes, sir.

T. What did you really do when you added that $-a$ to $5a$?

P. We subtracted it from $5a$.

T. If you added another $-a$ to $4a$, how would the sum compare with $4a$?

P. It would be smaller.

T. How much smaller?

P. a smaller.

T. We would generally write it $3a$, but we might write it $4a - a$; wouldn't that be equal to $3a$?

P. Yes, sir.

T. Well, I want to add $-b$ to $4a$, how should the sum compare with $4a$?

P. It should be smaller.

T. How much smaller?

P. b smaller.

T. What is the quantity b smaller than $4a$?

P. $4a - b$.

T. Then that must be the sum of $4a$ and $-b$. Take another illustration. To $3d + 2x$ add $-x$.

P. (Doing the example.) $3d + x$.

T. How much smaller than $3d + 2x$?

P. x smaller.

T. Add another $-x$.

P. (Pupils do so.) $3d$.

T. How much again is your quantity diminished by adding $-x$? P. x.

T. If you add another $-x$, how much smaller ought the sum to be than $3d$? P. x smaller.

T. What quantity is x smaller than $3d$?

P. $3d-x$.

NOTE. After a little more attention to this point, the drill, using the series, should continue, the pupils adding as the teacher points out the numbers. Have pupils write answers.

DRILL 11.

TEACHER. (POINTING OUT.)	PUPILS ANSWER.
Add $5a$ and $3a$	$8a$
$5a$ and $3b$	$5a+3b$
$5a$ and $-3b$	$5a-3b$
$3ab$ and 6	$3ab+6$
$3ab$ and -6	$3ab-6$
$6x$, $3b$, and $2c$	$6x+3b+2c$
$6x$, $3b$, and $-2c$	$6x+3b-2c$
$6x$, $-3b$, and $-2c$	$6x-2b-8c$
etc.	

T. Now, as I point out like terms, write them in the same column.

P. (Raising his hand.) What are like terms?

T. Like terms are terms that have the same literal factor. For example, $5a$, $7a$, $9a$ are like terms because the literal part is alike in all. So $5ab$, $3ab$, $4ab$ are like terms. $7x^2$, $12x^2$, etc., are like terms; but $7x^2$ and $6x$ are not like terms; for, though the letter is the same, the power is different, hence the literal factor is different. In addition and subtraction, you can always unite like terms into a single term by adding or subtracting the numerical coefficient of the common literal factor. As, $7a$ and $5a$; the sum? P. $12a$.

T. $8ab$ and $6ab$; the sum?

P. $14ab$.

T. $6a$ and $9ab$; the sum?

P. $6a + 9ab$.

T. Why couldn't you unite those into one term?

P. Because they were not like terms.

T. All numerical terms, that is, those having no letter in them, are also like terms, as 6, 9, 20, etc. But you will learn this by doing. Write down the quantities as I point them out, adding them as you do so whenever you can.[1] $6a$, $3b$. (The pupils write $6a + 3b$, and so on.) $-3d$, $3a$,—first, pupils, see if you have any term already like this.

P. Yes, sir; the first one, $6a$.

T. Then write the $3a$ under it. (The pupils do so.) -5, have you any term like that?

P. No, sir.

T. Then add it after the last term in the upper line. (The pupils do so.) $-3x^2$, $7b$; have you any term already like that?

P. Yes, sir; the second term.

T. Then write it under the second term, and be careful to put the proper sign before it. Every term you write should have either a plus or minus sign before it except the positive terms in the first column. (After seeing that $7b$ is written correctly.) $4x^2$, $3x$, can this be written in the column of x^2?

P. No, sir.

---

[1] The teacher had better write on the board as the explanation goes on, letting the pupils imitate till they get a good form.

T. Then we will add it as we have other new terms. (Doing so). $5d$, $-4a$, $-6x^2$, -7, $3d$, -8, $-5b$, $-4d$. That will do; now let us add all together.

Note. The above example will now be in this form. In adding begin at the left.

$$\begin{array}{lrrrrrr} & 6a & +3b & -3d & -3x^2 & +3x & -7 \\ & 3a & +7b & +5d & +4x^2 & & -8 \\ & -4a & -5b & +3d & -6x^2 & & \\ & & & -4d & & & \\ \cline{2-7} \text{Added:} & 5a & +5b & +d & -5x^2 & +3x & -15 \end{array}$$

T. In the same way write down the following quantities as I point them out. (Send three or four of the slower pupils to write on the board, changing them after each example.)

Drill 12.

1. $3a$, $4b$ $-2c$, $-8a$, 7, $5c$, $-3b$, -6, $4x^2$, $-3c$, $5a$, $6b$.
2. $5x^2$, $-4x^2$, $3x$, $-4x^2$, $-7x$, -9, $3x^2$, 12, 7.
3. $2ab$, 7, $5ab$, -8, $6x$, $-4ab$, $-3x$, $-3x$, 4.
4. $4x^2$, $-6ab$, $7a$, 9, $-8x^2$, $4ab$, $-3a$, 5, x^2, a.

Note. After doing this example, have the pupils rearrange the terms of the answer in the order of the letters.

This drill can be continued as long as desirable, but leave time for the following problem.

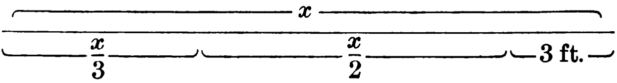
T. Now, scholars, I want to give you a problem a little different from any that you have had.

A pole stands one-third its length in mud, one-half in water, and three feet out of the water; how long is the pole?

Tell me right off; don't stop to compute.

P. x.

T. Yes. Now can you tell me what else x equals? for you know to form an equation we must always have two quantities that are equal. (As the pupils hesitate, the teacher draws the pole, and gets from the pupils the parts of it as follows: —)



T. Now, scholars, what does the sum of these parts equal?

P. The length of the pole.

T. What did you say the length of the pole is?

P. x.

T. Then let us put that into the form of an equation. I will write $x =$, and I want some one of you to write the other member. (After a little study most of the pupils offer to try, and the following equation is formed: —)

$$x = \frac{x}{3} + \frac{x}{2} + 3.$$

T. How can we eliminate the denominator 3?

P. By multiplying the equation by 3.

T. How can we eliminate the 2?

P. By multiplying the equation by 2.

T. Must we multiply first by 3 and then by 2, or can we multiply by some number that will eliminate the 3 and 2 at once?

P. (Studying a moment.) Multiply by 6. (If any say 5, show them their error by multiplying by 5.)

T. Let us do so.

$$x = \frac{x}{3} + \frac{x}{2} + 3$$
$$6x = 2x + 3x + 18$$
$$x = 18.$$

T. One more, scholars.

Two-thirds of a boy's money is in nickels, and two-sevenths in dimes, and four cents in one cent pieces; how much money has he? and how much in nickels and dimes each?

Tell me at once how many cents he had in all.

P. x cents.

T. I think that is right; let us prove it. How many of them were in nickels?

P. Two-thirds of them.

T. Write it. (Pupils write $\frac{2x}{3}$.) How many of them in dimes?

P. Two-sevenths of them.

T. Express it, and add it to the other. (The pupils do so.) How many cents more did he have to make up the whole?

P. Four cents.

T. Add them on. (The pupils do so.) What, then, is the whole number of cents equal to?

P. x.

Note. The equation is then formed and left for the pupils to work out.

LESSON VIII.

Subtraction — Negative Quantities — Miscellaneous.

NOTE. Begin the lesson with a brief drill in addition. Then change to Drill 5, p. 44, ending thus: —

T. Which is the larger?	PUPILS.
$18a$ or $12a$?	$18a$; $6a$ larger.
$-18b$ or $-12b$?	$-12b$; $6b$ larger.
$9x$ or $7x$?	$9x$; $2x$ larger.
Subtract $7x$ from $9x$.	$2x$.
Subtract $9x$ from $7x$.	(Pupils hesitate.)
Well, let us see.	

TEACHER.	PUPILS (ANSWERING).
$3a$ from $7a$, what?	$4a$.
$4a$ from $7a$, what?	$3a$.
$5a$ from $7a$, what?	$2a$.
$6a$ from $7a$, what?	a.
$7a$ from $7a$, what?	0.

T. Why did you get one a less each time?

P. Because we subtracted one a more each time.

T. If you should subtract one a more still, how much smaller ought the remainder to be?

P. a smaller.

T. What is the quantity a smaller than 0?

P. Minus a.

T. Then $8a$ from $7a$ ought to leave what?

P. $-a$ (writing it).

T. Certainly. If I had 7 dollars and spent 8, how would I stand financially? In arithmetic, I would have to say that I was one dollar in debt, but in algebra I can say that I have minus one dollar; that is, I have a dollar less than none.

QUESTIONS CONTINUED.	ANSWERS BY PUPILS.
$9a$ from $7a$, what?	$-2a$.
The larger, $12a$ or $5a$?	$12a$, $7a$ larger.
$5a$ from $12a$?	$7a$.
$12a$ from $5a$?	$-7a$.
The larger, $3x$ or 0?	$3x$, $3x$ larger.
0 from $3x$?	$3x$.
$3x$ from 0?	(hesitating a little) $-3x$.

T. Let us count it off in the series of x's (which has been kept on the board). In adding, you remember, we counted positive quantities upward, and negative quantities downward. In subtraction we should count positive quantities downward, because they diminish the minuend. Starting from zero, therefore, and counting off three x's, it gives us one, two, three, to minus $3x$, as you have said.

(A pupil is now sent to the board, and verifies each of the following examples by counting.)

T. Subtract $5x$ from $9x$.	P. $4x$.
T. Subtract $8b$ from $4b$.	P. $-4b$.
T. Subtract $3a$ from a.	P. $-2a$.
T. Subtract $9c$ from 0.	P. $-9c$.
T. Subtract $18ab$ from ab.	P. $-17ab$.
T. $-6b$ from $3b$.	(Pupils give various answers.)

T. Why, scholars, what is the difference between $-6b$ and $3b$; which is the larger?

P. $3b$; $9b$ larger.

T. Then what ought I to have left if I take $-6b$ from $3b$?

P. (Hesitatingly.) $9b$?

T. I see you do not feel sure. Why do I have two left if I subtract 7 from 9?

P. Because 9 is two larger than 7.

T. Why do I get 6 when I subtract 1 from 7?

P. Because 7 is 6 larger than 1.

T. Then what should be left if we subtract $-6b$ from a number $9b$ larger than itself?

P. (More confidently.) $9b$.

T. It seems reasonable, doesn't it? You have learned in addition that adding on a positive quantity gives a sum larger than the quantity to which it is added, and that adding on a negative quantity has just the opposite effect, that is, gives a sum smaller than the quantity to which it is added. Now you know that subtracting a positive quantity leaves a remainder less than the minuend; what, then, ought to be the effect of subtracting a negative quantity?

P. It ought to leave a remainder larger than the minuend.

T. That seems reasonable too. In subtracting a positive quantity by counting off, we count downwards; how ought we, do you think, to count in subtracting a negative quantity?

P. Upwards.

T. Then beginning at $3b$, let us count upward $6b$. (This is done, giving $9b$ as before.)

But, scholars, let us test it another way. From $4a$ subtract each of the quantities I give you, keeping the examples side by side. First, $4a$ itself; next $3a$, $2a$, a, 0. Now, scholars, as you have subtracted quantities decreasing from $4a$ to 0, how has it affected the remainders?

P. They have increased from 0 to $4a$.

T. If we kept decreasing the subtrahend, what ought the remainder to continue to do?

P. To increase.

T. Then subtract $-a$ from $4a$.

P. $5a$.

T. $-2a$ from $4a$.

P. $6a$. (These may be verified by counting.)[1]

T. $-3a$ from $4a$. P. $7a$.

T. $4a$ from $-3a$. (The pupils hesitate.) Well, let us go back. (Have the pupils write out these examples.) 7 from 12; 12 from 7; 9 from 10; 10 from 9; $5a$ from $8a$; $8a$ from $5a$; 0 from $3x$; $3x$ from 0.

Now, scholars, tell me why some of your remainders have the positive, and some the negative sign?

NOTE. After studying the examples, the pupils come, one by one, to see the principle and state it thus:—

When a smaller (by a positive quantity) is subtracted from a greater, the remainder is positive.

When a greater is subtracted from a smaller, the remainder is negative.

---

[1] Do not make too much of the counting process. It may serve to illustrate, but is artificial, as has been said before.

The teacher then has the pupils rapidly do the following examples, putting in each case the question, "Are you subtracting a less from a greater, or a greater from a less?"

Subtract 5 from 10; 10 from 5.
Subtract -5 from -10; -10 from -5.
Subtract $-6a$ from $4a$; $4a$ from $-6a$.
Subtract 0 from $5x$; $5x$ from 0.
Subtract 0 from $-5x$; $-5x$ from 0.
Subtract $4ab$ from 0; $-4ab$ from 0, etc.

Drawing a table like this:

No.	1st Quantity.	2d Quantity.	Sum.	1st − 2d.	2d − 1st.
1	$8a$	$3a$	$11a$	$5a$	$-5a$
2	$5x$	0	$5x$	$5x$	$-5x$

The teacher inserts two quantities in the 2d and 3d columns, and has the pupil fill out the other three as indicated above.

T. (Resuming the drill.) a from $3a$ leaves what?
P. $2a$.

T. $-a$ from $3a$ leaves what?
P. $4a$.

T. b from $3a$ leaves what?
P. $3a - b$.

T. $-b$ from $3a$ leaves what?

Note. The pupils will readily see that it must be $3a + b$. It can also be shown by counting from $-b$ up to zero on the b series, and from zero up to $3a$ on the a series. Fortify the fact with a lot of drill, especially in subtracting positive and negative quantities from zero, thus: 7 from 0; -7 from 0; $3a$ from 0; $-3a$ from 0; $5ab$ from 0; $-5ab$ from 0; etc.

T. So far, scholars, we have been dealing in subtraction with monomials; but we may also have to subtract polynomials. For example: from $2a + b$ take $a + b$. (The pupils write this down and subtract properly, as also the following)[1]: —

T.

From $a + b$ subtract $a - b$. (Also add.)
From $x + 4$ subtract $x - 4$. (Also add.)
From $7a^2 - 3a$ subtract $9a^2 - 7a$. (Also add.)
From $4b + 8a^2$ subtract $9b + 10a^2$.
From $4 - a$ subtract $8 - 2a$.
From $x + y$ subtract $x - y$.
From $x - y$ subtract $x + y$. (Also add.)
From $x + y$ subtract $-x - y$. (Also add.)
From $2a + b - e$ subtract $a + b - e$.

From $4x + 2y - 4$ subtract $3x - 3y - 9$.

From $4a^2x - ax^2 + 2bx$ subtract $2a^2x - ax^2 - 2bx$.

From $5a^2 - 3ax + 9x^2 + 4$ subtract $2a^2 + 5ax - 3x^2 + 4$.

From $7b + 4ac - 3ad - 5dc$ take $7b + 2ad + 3b^2 - 4$.

Note. Show the pupils how to write this in some orderly way, as, for example: —

$$\begin{array}{llllll} 7b & + 4ac & - 3ad & - 5dc & & \\ \underline{7b} & \underline{} & \underline{+ 2ad} & \underline{} & \underline{+ 3b^2} & \underline{- 4} \end{array}$$

Then, as $7b$ from $7b$ leaves 0, it may be left blank;

---

[1] In subtraction *do not* change the signs of the subtrahend and add; and do not "suppose them changed." This device prevents the pupil from seeing the real nature of the operation, and at this stage of progress is an injury. But go through every third or fourth example term by term, and have the pupil tell whether he is subtracting a less from a greater, or a greater from a less.

0 from 4 ac leaves 4 ac; 2 ad from $-3\,ad$ leaves $-5\,ad$; 0 from $-5\,dc$ leaves $-5\,dc$; 3 b^2 from 0 leaves $-3\,b^2$; and -4 from 0 leaves 4.

A little more drill like the following may be well at this stage: —

From 3 x take 0; from 0 take 3 x.
From -5 take 0; from 0 take -5.
From 7 a take 0; from 0 take $-4\,ab$.
From 0 take 8 x; from 0 take $-3\,xy$.
From 0 take 2 ab; from 0 take $-7\,ab$, etc.

From a^2+b^2 take $ab+b^2$; from a^3+b^3 take $a^3-b^2a+ab^2$; from a^4+b^4 take $4\,a^3b-6\,a^2b^2+4\,ab^3$.

T. Permit me, scholars, at this point to explain the theory of this.

Every quantity is measured from zero as an origin; that is, it is so much greater or so much less than zero, or is zero itself. For example: a means a quantity a greater than zero, and $-a$, a quantity a less than zero. Whether we think of it or not, all quantities carry with them this idea, "greater than," or "less than" zero, as if written: $\mathbf{0}+a$, or $\mathbf{0}-a$. So, too, polynomials; only they are made up of elements, each of which is greater or less than zero. For example: the quantity $a-b-c$ must be taken as a whole made up of elements, one of which tends to make it greater and the other two less than zero. The signs show the relation of each element to the origin zero, as if written, $\mathbf{0}+a-b-c$. Now, if I want to add to this a quantity unlike any now in it, say d, as I can't combine it with one of its present terms, I add it as a new element greater than zero, $a-$

$b - c + d$. If I had wanted to subtract the d, I would have subtracted it by sign from the origin 0, getting $a - b - c - d$

NOTE. Continue the drill.

T. I have a sum of money; if I had 10 dollars more, I should have 6 dollars; how many dollars have I? The result is $x = -4$.

T. Scholars, how do you interpret the negative result? By "interpreting the negative result," we mean explaining it in the language of arithmetic, which, you know, does not recognize negative quantities. For example: if I got as an answer to an algebraic problem, — 5° north, I would interpret it to mean 5° south. How do you interpret this answer, giving me minus four dollars?

P. You are four dollars in debt, etc.

NOTE. Hereafter the special problems introduced into the lessons will not be explained, but will be left for the teacher to develop. Whenever any peculiar feature is introduced into the problems, a sample problem will be given in the corresponding lesson of the Guide.

LESSON IX.

A General Review.

NOTE. Use this lesson for drill, and for clearing up any points where the pupils are weak or in doubt. Of what is needed the teacher should be the best judge.

PROBLEM. The sum of two numbers is 50, and the difference 10; what are the numbers?

LESSON X.

Multiplication by Negative Quantities.

NOTE. **Begin with a brief drill in addition and subtraction.**

TEACHER. Multiply a^5 by a^2. PUPIL. a^7.
Multiply $3x^2$ by $2x$. $6x^3$.
Multiply $7a^2b$ by $2ab^3$. $14a^3b^4$.
Multiply a^3 by $-a^2$. (Some of the pupils say a^5 and some are doubtful.)
Multiply a^3 by a^2. a^5.

T. Is a^2 and $-a^2$ the same quantity?

P. No, sir.

T. Then, will a^3 multiplied by $-a^2$ be the same as a^3 multiplied by a^2? (Pupils think not.) The sum of the exponents is 5, is it not?

P. Yes, sir.

T. Will it be a^5 or $-a^5$? (The pupils think $-a^5$, but are uncertain.) Well, scholars, we have now met a very important question in multiplication. You already know something about multiplication. You know that multiplying a positive quantity by a positive quantity, as a by 6, or b by e, gives a positive product. What is 6 times a?

P. $6a$.

T. That is, if a is greater than zero, 6 times a must be greater than zero. (Pupils are made to see this.) Now, the question is, what must be the sign of the product if a negative and a positive are multiplied together, or if two negatives are multiplied together. Let us attack the problem. Suppose a man had -5 dollars, which, interpreted, means that he was five dollars in debt, and another had three times as much, would you understand that the latter was out of debt and ten dollars over, or three times as deep in debt?

P. Three times as deep in debt.

T. How much would you say he had, calling it assets?

P. — 15 dollars.

T. Then, as a matter of common-sense, you would say that — 5 multiplied by 3, gives — 15, which would seem to indicate that a positive and a negative quantity multiplied together give a negative product.

P. Yes, sir.

T. We will try it another way, however, and see if this is confirmed. What will a multiplied by 4 give?

P. $4a$.

NOTE. The teacher continues this till the following series of examples are on the board:—

$$\begin{array}{c} a \\ 4 \\ \hline 4a \end{array} \qquad \begin{array}{c} a \\ 3 \\ \hline 3a \end{array} \qquad \begin{array}{c} a \\ 2 \\ \hline 2a \end{array} \qquad \begin{array}{c} a \\ 1 \\ \hline a \end{array} \qquad \begin{array}{c} a \\ 0 \\ \hline 0 \end{array}$$

T. As I have multiplied a by a number one less each time, what has been the effect on the product?

P. It has decreased by a each time.

T. Then, if I should multiply a by a quantity one less than zero, how ought the product to compare with the product of a multiplied by zero?

P. It ought to be a less or $-a$. (See that every pupil understands this, even if you have to go over the above examples time and again.)

T. Let us try it. What is the multiplier one less than zero?

P. -1.

T. $\left(\text{Writing on the board } \begin{array}{r} a \\ \underline{-1} \end{array}\right)$ And the product a less than the product zero?

P. $-a$.

T. Which confirms your previous judgment. I think, scholars, we can now act on the law that a negative and a positive quantity multiplied together gives a negative product.

T.	Multiply	$-x$	by	6.	P.	$-6x$.
	Multiply	$-a$	by	b.		$-ab$.
	Multiply	a	by	$-b$.		$-ab$.
	Multiply	x	by	-6.		$-6x$.
	Multiply	$-3ab^2$	by	a^2b.		$-3a^3b^3$.
	Multiply	-7	by	8.		-56.
	Multiply	-7	by	-3.	(Pupils hesitate again.)	

T. This brings us to the second part of our question: what must be the sign of the product if two negative quantities are multiplied together? Let us first multiply $-a$ by 4 (putting these examples on the board and writing the answers obtained from the pupils); next by 3; next by 2. Let me call your attention here to the fact that a negative quantity is of such a nature that the fewer you have of them the more you have. You would be better off to have only two times -5 dollars than to have four times -5. Now, as I have multiplied $-a$ by a quantity one less each time, how has it affected the product?

P. It has become a larger.

T. Let us keep on; $-a$ by 1; what will it be?

P. $-a$.

T. If I multiply by one less than 1, or by 0, how must the product compare with $-a$?

P. It must be a larger, or zero (the example is added to the others).

T. And now, scholars, if I multiply $-a$ by one less than zero, or -1, how ought the product to compare with zero?

P. It ought to be a larger, or a. (Draw this answer from the pupils, if you have to review the above examples a dozen times.)

T. If I multiply $-a$ by -2, what then ought it to give me?

P. $2a$ (continue as long as necessary on this point).

T. Then, scholars, I think we may now act on the principle that two negative quantities multiplied together give a positive product. It all rests on the nature of a negative quantity, which, as I have said, is such that the fewer you take of them the larger the product; if you take them more than zero times, you get a product less than zero; if you take them zero times, of course you get zero; and if you take them less than zero times, you get a product greater than zero.

But, scholars, I want to test this another way. What really is multiplication?

P. A short method of addition. (This should be made clear to all the class.)

T. Instead of multiplying 7 by 5, how otherwise can we find the product?

P. By adding together 5 sevens. (They do it.)

T. You speak of adding the sevens together; I would rather have you think of it as adding them to zero, thus: (see example). The reason for this will appear later. Let me, then, give you this definition of multiplication, which will be rather more complete than the one you gave. Multiplication is a short way of adding a quantity any number of times to zero. For example, multiplying a by 4 means adding a to zero four times, thus: (see example, which should be made clear to all the pupils). Now, scholars, let us test this further and see what comes of it. Let us find in this way the products of a multiplied by 3, 2, and so on, thus (have the pupils do the work):—

$$7 \times 5 = \begin{array}{r} 0 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ \hline 35 \end{array}$$

$$a \times 4 = \begin{array}{r} 0 \\ a \\ a \\ a \\ a \\ \hline 4\,a \end{array}$$

$$a \times 3 = \begin{array}{r} 0 \\ a \\ a \\ a \\ \hline 3\,a \end{array} \qquad a \times 2 = \begin{array}{r} 0 \\ a \\ a \\ \hline 2\,a \end{array} \qquad a \times 1 = \begin{array}{r} 0 \\ a \\ \hline a \end{array} \qquad a \times 0 = \begin{array}{r} 0 \\ \hline 0 \end{array}$$

Note. The attention of the scholars is called to the fact that, as a is multiplied by a number one less each time, the product is a less.

T. (Continuing.) What should the product be as compared with zero if we multiply a by a number one less than zero?

P. It ought to be a less than zero, or $-a$.

T. How shall we prove it as we have done the others, by adding to zero? (The pupils try this, getting a.)

Scholars, I think the trouble comes in not interpreting correctly what adding a to zero *minus once* means. If I spoke of -4° north, how would you interpret it?

P. As meaning 4° south.

T. If I said I deposited -5 dollars in a bank, how would you interpret it?

P. That you withdrew 5 dollars.

T. What is the opposite of addition?

P. Subtraction.

T. How would you interpret adding minus once?

P. As subtracting once.

T. Yes. Do these examples.

Add 4 to 0 minus once.

P. -4.

T. Then, what will 4 multiplied by -1 be?

P. -4. (Repeat the above question after each of the following.)

T. Add 4 to 0 minus twice. P. -8.

Add -4 to 0 twice. -8.

Add 4 to 0 minus three times. -12.

Add -4 to 0 minus once. (The pupils may at first stumble on this, but finally get it 4.)

T. Scholars, let me impress upon you one caution: be careful to observe the signs when you are dividing; most of your mistakes will be through carelessness in this respect.

PROBLEM. A grocer sold the same number of pounds of coffee and tea: for the coffee he got 38 cents a pound and for the tea 76 cents a pound; for the whole he received \$11.40; how many pounds did he sell of each?

LESSON XI.

Multiplication continued. Powers of Quantities.

NOTE. Begin with a brief drill in simple multiplication.

TEACHER. Multiply $a + 4$ by $a + 3$. (The pupils hesitate.) Why, scholars, what is the trouble?

PUPILS. We can't multiply $a + 4$ by $a + 3$.

T. Why, then, multiply $a + 4$ by a. (The pupils do so, getting $a^2 + 4a$.) Now multiply $a + 4$ by 3. (The pupils do so, getting $3a + 12$.) Now add the products together. (The pupils do so, getting $a^2 + 7a + 12$.) Are you aware, scholars, that you have multiplied $a + 4$ by $a + 3$? (The pupils look surprised.) Let us go to work now a little more systematically and see how easy it is. First, multiply 327 by 49. (The pupils do this work all out, and the teacher calls attention to the fact that they multiplied by the figures of the multiplier separately and added the results.) This is just the way you do in algebra, except that instead of beginning with the last term of the multiplier, as you do with the last figure of the multiplier in arithmetic, it is better to begin by multiplying with the first term; better on account of writing the signs, as you will see.

Multiply $2x - 7$ by $x - 4$. (The teacher shows the pupils how to write them.) Now multiply the multiplicand by x; but first tell me what the sign will be, multiplying $2x$ by x.

P. Positive.

T. Then, as it is the first term in the product, you need not write the sign. What is x times $2x$?

P. $2x^2$. (They write it down.)

T. What terms shall we multiply together next? (Some think -7 by -4; but the teacher impresses it upon them that they had better finish multiplying by x, and that this is what they do in multiplying numbers. So it is finally decided to multiply -7 by x.) First, what will the sign be?

P. Negative (and they write after the $2x^2 - 7x$).

T. What shall we multiply by next?

P. -4.

T. Which term of the multiplicand shall we multiply by -4 first? (This is settled.) What must the sign of the product be?

P. Negative.

T. And the product, then?

P. $-8x$.

T. See if there is any like term in the product already found that you can place this with. (The pupils suggest $-7x$, and place the $-8x$ under it.) What shall we multiply next?

P. -7 by -4.

T. Look carefully at the signs, and tell me what the sign of the product will be.

P. Positive.

T. Then what will the product be?

P. 28.

T. Have you any similar term under which you can write it?

P. No, sir. (Teacher then shows them where to write it, and has them add the results, getting the product.)

T. You see, scholars, it is simply multiplying a quantity by several monomials, as you have been doing, and adding the products together, just as you do in arithmetic.

Note. In this same way multiply: —

$$\begin{array}{ll} x^2 - 2x - 8 & \text{by } x + 3. \\ a^2 - 5x + 6 & \text{by } 2a - 4. \\ x + 3 & \text{by } x - 3. \end{array}$$

T. Sometimes, in multiplying one quantity by another, scholars, the terms do not come so conveniently alike. For example: multiply $3a^3 - 4ab$ by $a - b$. (Show them how to do this example as below, how to write new terms, and how to arrange them according to the power of some letter. Emphasize care as to signs.)

$$\begin{array}{llllll} 3a^3 - 4ab + 2b^2 & & & & & \\ \underline{\;a \;\; - b\;} & & & & & \\ 3a^4 - 4a^2b + 2ab^2 & & & & & \\ \qquad\qquad + 4ab^2 - 3a^3b - 2b^3 & & & & & \\ \hline 3a^4 - 3a^3b - 4a^2b + 6ab^2 - 2b^3 & & & & & \end{array}$$

Multiply $x^3 - 2x + 5$ by $x^2 - 4$.

T. What does this mean (a^2)?

P. a square, or a multiplied by a.

T. And this, $(a + b)^2$?

P. The square of $a + b$.

T. The square of a plus b might mean $a^2 + b$. (After a little thinking by the pupils, the teacher explains that it should be read, the square of the quantity $a + b$, etc., and has the scholars expand the expressions by multiplication.)

Problem. I bought some knives for 21 cents each, and the same number of forks for 15 cents each; I also paid 10 cents for expressing them home; the whole cost was $5.50; how many knives and forks did I buy?

LESSON XII.

Division.

Teacher. (After a brief drill in multiplication.)

Divide a^2 by a;
Divide a^3b^2 by ab^2;
Divide $9x^3y^2$ by $3x^2y$;
Divide $36a^2b^4$ by $12a^2b^3$;
Divide b^4 by $-b$. (Some of the pupils say b^3.)

T. You did not notice, I fear, that the divisor I gave you was *minus* b. Divide b^4 by b.

Pupils. b^3.

T. Is b and $-b$ the same quantity? P. No, sir.

T. Then, do you think that b^4 divided by $-b$ ought to be the same as b^4 divided by b?

P. No, sir. (One of them.) I guess it is $-b^3$.

T. I am inclined to think that your guess is correct, but let us study the matter a little. We know that dividing a positive by a positive gives a positive quotient. What is the question of division that we must now settle? (Hands go up.)

P. What the sign of the quotient will be if we divide a positive quantity by a negative, or a negative by a positive.

T. One more condition illustrated by this (writes on the board): — 6)— 12.

P. Or a negative quantity by a negative.

T. Before beginning our investigation, it may be well to decide just what division means. What is division?

P. A process of finding how many times one number is contained in another. (All think so.)

T. Perhaps so. That is the common notion of division, but common notions may be wrong. What have we, in our last lesson, said that multiplication really is?

P. A short way of adding equal quantities.

T. Isn't division the opposite of multiplication?

P. Yes, sir.

T. Then, how should we define it.

P. (After a little stumbling.) As a short process of subtraction.

T. Yes. To illustrate, we ask: how many times 5 will go in 15; and this is well enough for practical purposes, as in arithmetic. But, for a scientific investigation, we must get down to the fundamental idea, which is subtraction. In other words, division is a short process for finding how many times one quantity can be taken out of another, leaving zero. For example: 5 in 15, 3 times. The quotient shows that 5 can be taken out of 15 three times, leaving zero, thus: —

$$\begin{array}{r}15\\ \underline{5}\\ 10\\ \underline{5}\\ 5\\ \underline{5}\\ 0\end{array}$$

Take another example. (The pupils do the examples as given, preserving the work.) Divide 8 by 2. (The pupils divide 8 by 2.) What does the quotient, 4, show?

P. That 2 can be subtracted 4 times from 8, leaving zero.

T. Prove it. (The pupils do so.) How many times less than 4 can we subtract 2 from a quantity 2 less than 8, before reaching zero?

P. Once less, or three times.

T. Prove it. (The pupils do so by subtracting.) Then, in dividing 6 by 2, what does the quotient 3 mean?

P. That 2 can be subtracted three times from 6, leaving zero.

(Repeat this for a quantity 2 less than 6, and 2 less than 4.)

T. How many times less than once, can we subtract 2 from a quantity 2 less than 2?

P. Once less, or no times.

T. What is the quantity 2 less than 2?

P. Zero.

T. Divide zero by 2. (The pupils write $\begin{array}{r}2\underline{)0}\\ 0\end{array}$.) What does the quotient 0 show?

P. That no times 2 from zero leaves zero.

T. Yes, and that would be true for any divisor. (Illustrate this for 3, 4, etc.) Now, suppose we subtract 2 from a quantity 2 less than 0, how many times less than no times can we do it?

P. Once less.

T. What is the quantity 2 less than zero?

P. – 2.

T. And the quotient 1 less than zero?

P. – 1.

T. Then, performing the example, it ought to be this (putting it on the board) $\begin{array}{r}2\underline{)\,-2}\\ -1\end{array}$, ought it not?

P. Yes, sir.

T. Prove it. (The pupils write it down thus: $\begin{array}{r}-2\\ \underline{2,}\end{array}$ but in subtracting get – 4.) That never will do. Division is a short process of subtracting one quantity from another till zero is reached, but you are getting farther from zero. Let us try another. If 2 goes in – 2 minus once, how many times less ought it to go in a quantity 2 less then –2? (Review if necessary.)

P. Once less, or – 2. (Doing the example they get $\begin{array}{r}2\underline{)\,-4}\\ -2\end{array}$, but in trying to prove it by subtraction, they get – 8.)

T. There must be something wrong, and I suspect it is because we don't quite understand what that negative sign of the quotient really means. Take the example, – 2 divided by 2 is – 1, and see if you can do anything with the dividend and divisor to get zero. (After a little study hands begin to go up.) Well, what have you found?

P. We can add 2 to -2 and get zero.

T. Try it with -4 divided by 2 is -2. (The pupils discover that by adding the divisor twice they get zero.) Now, scholars, I think you are ready to understand the principle; but first let me ask you some familiar questions. When we speak of north latitude *as* north latitude, what sign do we give it?

P. The positive sign.

T. When we speak of south latitude as north latitude, what sign do we give it?

P. The negative sign.

T. What kind of a deposit is a withdrawal?

P. A negative deposit.

T. What kind of a withdrawal is a deposit?

P. A negative withdrawal.

T. What kind of a subtraction is an addition?

P. (Beginning to see the point.) A negative subtraction.

T. That is, if I added a quantity to another twice, how many times might I say I subtracted it?

P. Minus twice.

T. And if I said I subtracted a quantity from another minus three times, what would it mean arithmetically?

P. That you added it three times.

T. Subtract 3 from 7 once. P. 4.
Subtract 3 from 7 minus once. 10.

Subtract 3 from 7 twice. 1.
Subtract 3 from 7 minus twice. 13.
Subtract 4 from 8 twice 0.
Then 4 will go in 8, twice.
Subtract 4 from — 8 minus twice. 0.
Then 4 will go in — 8 minus twice.

Now, scholars, the whole principle may be thus summed up: Division is a short process for finding how many times one quantity can be *subtracted* from another, leaving zero. If zero can be reached by *actual subtraction,* then the quotient is positive; but, if to get zero we must *add* instead of subtract, then we must call it subtracting a negative number of times; that is, the quotient is negative.

Let us take a few cases, and discuss them.

First, 18 divided by 6; tell me whether I must add the 6 to 18 three times, or subtract it three times, to get zero.

P. Subtract it three times.

T. Then, what will be the sign of the quotient?

P. Positive.

T. If both the dividend and divisor are positive, could you ever reach zero by adding?

P. (After a little thought.) No, sir.

T. What must you always do?

P. Subtract.

T. Then, what must the quotient always be?

P. Positive.

T. Next divide — 20 by 5. Now, scholars, before doing the example, tell me, if you find you can get zero by actually subtracting 5 from – 20, four times, what will be the quotient. P. 4.

T. But if you find you must *add* it four times, what will it be, remembering that division is *subtraction?*

P. – 4. (The pupils do the example, finding that they must add; and they write the quotient – 4.)[1]

T. If the dividend is negative and the divisor positive, could you ever reach zero by subtracting? (The pupils try, and find that they must always add, so that the quotient will be always negative.)

T. Next divide 15 by –3. Evidently, scholars, the quotient will be either plus five or minus five. If you can get zero by subtracting, which will it be?

P. Plus five.

T. If by adding?

P. Minus five. (The pupils find they have to add, and give the answer – 5.)

T. If the dividend is positive and the divisor negative, could you ever reach zero by subtracting? (This is settled as before.)

T. Now, scholars, there is one more combination. Divide – 21 by – 7. Again tell me, if you get zero by adding – 7 to – 21, what will the quotient be?

P. Minus three.

T. And if by subtracting? P. Plus three.

NOTE. The pupils find that they must subtract to get zero, and that they must always subtract when the divisor and dividend are both negative. From these examples they can draw the conclusion that like signs give a positive quotient and unlike signs a negative quotient.

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[1] The teacher will do well to ask what these answers really mean.

T.	Divide a^3 by a^2.	P.	a.
	Divide a^3 by $-a^2$.		$-a$.
	Divide $-a^3$ by a^2.		$-a$.
	Divide $-a^3$ by $-a^2$.		a.
	Divide $6a^3b$ by $-3ab$.		$-2a^2$.
	Divide $-25x^2y^2$ by $-5xy$.		$5xy$.
	Divide $6a-3b$ by -3		$b-2a$.
	Divide $18a^3b^2-18a^2b^3$ by $18a^2b^2$		$a-b$.
	Divide $-3a^2+6ab-b^2$ by -1, etc.		

Problem. 80 divided by a certain number is 1 more than 65 divided by the same number; what is the number?

LESSON XIII.

Division continued.

Teacher. Divide $8a$ by $2a$. (The pupils give 4 as the quotient.) What does the quotient 4 mean?

Pupils. It means that $2a$ can be subtracted out of $8a$ four times, leaving zero.

T. Prove it. (The pupils do the example as shown.) Divide $18x$ by $-6x$. (Pupils do example, getting -3.) What does the minus three mean?

P. That $-6x$ can be subtracted out of $18x$ minus three times.

T. All prove it. (This they do.) Interpret subtracting a negative number of times.

P. It means adding that number of times.

$$\begin{array}{rr} & 8a \\ 1, & \underline{2a} \\ & 6a \\ 2, & \underline{2a} \\ & 4a \\ 3, & \underline{2a} \\ & 2a \\ 4, & \underline{2a} \\ & 0 \end{array}$$

Note. Prove as above the following examples.

T.	P.
$-27b$ by $-9b$;	3.
$-12c$ by $3c$;	-4.
0 by 4;	0.
4 by 0;	0.

Please show me your work for this last example. (The pupils all have it $\begin{array}{l}0\underline{)4}.\\ \;\;\;0\end{array}$) Do you mean to tell me that zero times zero from 4 leaves zero? Subtract zero no times from 4 and prove your work. (The pupils see the dilemma.) Take these problems that I put on the board, scholars, write the quotient, and prove each of them by subtracting: —

$4\underline{)12}$, $3\underline{)12}$, $2\underline{)12}$, $1\underline{)12}$, $0\underline{)12}$, $-1\underline{)12}$, $-2\underline{)12}$.

NOTE. **Pupils find all the divisions can be proved except $0\underline{)12}$.**

T. What number of times do you think you would have to subtract 0 from 12 to get a zero remainder?

P. I don't think there is any number of times that would do it. Taking nothing from 12 doesn't make it less.

T. Then, you think it impossible?

P. Yes, sir. (All think so.)

T. Then, why not write it so, thus?

$$\begin{array}{l}0\underline{)12\qquad\qquad\qquad}\\ \quad\text{An impossibility.}\end{array}$$

It would be the exact truth. There are some very wise men who write the quotient 0 as you did before you thought, and then they try to explain it in a very strange way for wise men. They will tell you that the divisor zero does not mean exactly zero, but a quantity infinitely small, that is, too small to be expressed; and that the quotient does not at all mean zero, but a number of times infinitely great, that is, too large to be expressed, just the opposite, you see. If the quotient were infinity, it would be better to write it thus, ∞; for this is the symbol for infinity. But the quotient is not an infinitely large number. The divisor is zero, and an infinite number of times zero is no larger than seven or eight times zero is. The problem is simply an impossibility, as you have agreed, and should either be so stated, or else the place for the quotient should be left blank.

But to change the work, we have up to this time been dividing by monomials. We now must learn to divide by polynomials.

Divide $a^2 + 2\,ab + b^2$ by $a + b$. (As the pupils do not know how to proceed, the teacher puts it on the board as below.)

$a + b)a^2 + 2\,ab + b^2($ As the scholars still hesitate, they are asked to divide 1512 by 36, which the teacher has them do thus:

$$\begin{array}{r@{}l}
36)1512 & \left(\begin{array}{l}40 + 2 \\ \underline{\ \ 2\ \ } \\ 42\end{array}\right. \\
\underline{1440} & \\
72 & \\
\underline{72} & \\
0 &
\end{array}$$

T. We cannot tell at a glance how many times 36 will go in 1512, so we divide as near as we can. We say 36 in 1512, 40 times. We then multiply 36 by 40, and subtract from 1512 to see if it leaves zero, that is, we subtract forty 36's. We find we have a remainder 72; so we divide the remainder by 36 and say two times, which we add to the forty. Subtracting two 36's from 72 we reach zero, and so know that the quotient is 42.

Now try this on the algebraic example I have given you. How many times will the first term of the divisor go in the first term of the dividend? P. a times.

T. Write it as the first term of the quotient. What should we do next?

P. We should subtract a times a from the dividend.

T. a is not the divisor. It is $a + b$. You did not, in this example, subtract only 40 times 30, you subtracted 40 times 36.

P. We should subtract a times $a + b$.

T. Well, do so; you know how to multiply $a + b$ by a, I trust. (The pupils multiply $a + b$ by a, getting $a^2 + ab$ which they subtract, leaving $ab + b^2$.)

T. Evidently a isn't the whole quotient, for we have a remainder. How many times will a of the divisor go in ab, the first term of the remainder?

P. b times.

T. Well, add that to a, and subtract b times $a + b$, and see what it gives you. (The pupils do this, getting zero for remainder.) Then, how many times will $a + b$ "go in" $a^2 + 2\,ab + b^2$?

P. $a + b$ times.

Note. **In this same way work out the following:—**

$2\,x^2 - ax - 6\,a^2$ **divided by** $x + 2\,a$.

$3\,x^2 - 19\,x + 28$ **divided by** $3\,x - 7$.

$a^3 - 3\,a^2b + 3\,ab^2 - b^3$ **divided by** $a - b$.

$a^4 - 4\,a^3b + 6\,a^2b^2 - 4\,ab^3 + b^4$ **divided by** $a^2 - 2ab + b^2$.

PROBLEMS.

A father's age is five times his son's, but in 15 years he will be only twice as old; how old is each?

$\frac{52}{x} - \frac{39}{x} = 1$, to find the value of x.

LESSON XIV.

Division continued — Square and Cube Roots of Monomials — The Double Sign ± — The Radical Sign √.

TEACHER. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
Divide $a^2 - b^2$ by $a - b$.

NOTE. When the pupils reach this point: —

$$a - b)a^2 - b^2\,(a$$
$$\underline{a^2 - ab}$$

some of them may hesitate; if so, it may be well to review the theory of algebraic quantities in Lesson VIII.

See that the work is done in the proper order.

T. Divide $a^3 + b^3$ by $a + b$.
Divide $a^3 - b^3$ by $a - b$.
Divide $x^5 + y^5$ by $x + y$.
Divide $x^5 - y^5$ by $x - y$.

Divide $x^4 + y^4$ by $x + y$. $\left(\text{Last term} + \dfrac{2x}{x+y}.\right)$

Divide $x^3 - y^3$ by $x + y$. $\left(\text{Last term} - \dfrac{2x}{x-y}.\right)$

NOTE. After a suitable drill in dividing, change to squaring and to extracting the square root of monomials. First square the quantities by multiplication in full, having the problems put on the board and kept for comparison. If the pupils do not know what squares and square roots are, they should be taught.

T. Square 3; -3; a; $-a$; a^3; x^5; $-x^2$; a^2x; $-a^2x^2$. Now, scholars, I want you to look these examples through carefully, and tell me the relation of the exponents of the squares of the quantities I gave you to the exponents of the quantities themselves.

P. (After a careful study.) The exponents of the squares are twice the others.

T. Then how, without stopping to perform the multiplication, can you square a literal quantity?

P. By doubling the exponent.

T. Then, square ab; a^2b^2; x^2y, etc. If I gave you a literal quantity, how could you at once get the exponent of the square root?

P. By dividing the exponent by 2.

T. What is the square root of a^4?

P. a^2.

T. Prove it. (The pupils multiply a^2 by a^2, getting a^4.) Now multiply $-a^2$ by $-a^2$. (The pupils do so, getting a^4.) Then, how do you know whether the square root of a^4 is a^2, or $-a^2$, as either multiplied by itself, gives a^4? (The pupils confess they cannot tell.)

Note. Find the square root of two or three other quantities, as x^6, x^2, in the same way, and then explain that the square root may be either positive or negative, and, therefore, is usually expressed with a double sign, thus, $\pm x$, etc.

T. What is the square root of $-x^4$? (Some of the pupils, perhaps all, write $-x^2$.) Multiply $-x^2$ by itself and see what it gives you. (The pupils do so, getting x^4. Some may write $-x^4$, but of course their attention will be called to the error.)

Note. After the pupils have puzzled over this long enough, explain to them that the square root of a negative quantity cannot be extracted; hence, it is expressed by a sign, thus, $\sqrt{-a^2}$, and is called an imaginary quantity.

T. Square $4a^2$; ax^2; $2x^2y$; $5x^2y^3$.

Square root of $4a^2$; $9a^2x^2$; $16a^4x^2$; $81a^4b^2$?

Note. In the same way have the pupils learn inductively how to cube quantities and extract the cube root; observing that a cube root has not the double sign, and that the cube root of negative quantities can be extracted.

Problem. A has 20 bushels of apples more than B has, and five times what A has equals seven times what B has; how many has each?

LESSON XV.

Involution — The Binomial Theorem begun (for more difficult exercises, see Lesson XXXII.).

Note. Give the pupils some drill in expanding such quantities as $(3\,a^2x)^3$; $(-2\,x^2y^2)^5$, etc.; and also in extracting the root of such as $\sqrt{4\,a^2b^4}$, $\sqrt[3]{64\,a^3}$, $\sqrt[3]{-27\,a^6x^3}$, etc.

Then have them expand by multiplication, $(a \pm b)^2$, $(a \pm b)^3$, $(a \pm b)^4$, and $(a \pm b)^5$; half of the class taking $a + b$, and the other half $a - b$. Let them study till they see for themselves the law of exponents, and let them puzzle over it a little while before you show them the way to find coefficients.

Give them plenty of drill with such binomials as $a + 2$, or $x + 3$.

Do not let them write out all the literal factors of the terms first, and then put in their numerical coefficients afterwards, but have them write the terms complete as they go along. $(a + 3)^3$, however, may be first written $a^3 + 3\,(3)\,a^2 + 3\,(3)^2\,a + 27$, and simplified afterwards.

LESSON XVI.

Relation of Quantities — The Parenthesis — Factoring.

Note. Begin the lesson with a drill requiring the pupils to observe and express the relation of a quantity to one or two other quantities; that is, they should give the operation to be performed on one or two quantities to produce another. These operations, as far as the exercises below call for them, will be five in number: 1, A quantity may be the product of two others, or twice the product, etc.; 2, It may be the sum, twice the sum, etc.; 3, It may be the difference, twice, etc.; 4, It may be the square; 5, It may be the square root. The examples in Lesson XVI. of the algebra may be used for drill. The following illustrations will make clear what is required.

Teacher. What is the relation of ab to a and b?
Pupils. ab is the product of a and b.

T. Of $-3b$ to $5b$ and $-8b$?
P. $-3b$ is the sum of $5b$ and $-8b$.

T. Of $-35a$ to -7 and $5a$?
P. $-35a$ is the product of -7 and $5a$.

T. Of $a+b$ to a and b?
P. $a+b$ is the sum of a and b, etc.

The drill should be conducted orally, and will be valuable as a preparation for factoring.

T. Scholars, in writing algebraic quantities, it is often convenient or necessary to enclose two or more terms in a parenthesis, to show that the same operation must be performed on them or by them.

Thus, six times the quantity a plus b is written in this way (writing on the board), $6(a+b)$, and indicates that both a and b are to be multiplied by 6, making $6a + 6b$, if the parenthesis is removed. So $a(x + y) = ax + ay$.

The simplest use of the parenthesis is in factoring, so we will begin with factoring, and in that way learn to use it.

You have already, in arithmetic, learned something about factoring, yet it may be well to call attention to one or two points. What are the factors of 6?

P. 2 and 3 (2×3).

T. Yes; and yet in truth every number is a factor of 6. For example, 9 multiplied by $\frac{2}{3}$ gives 6. (Illustrate this by a few examples.) So you see all possible factors of any quantity would make a pretty large list. But by common consent, factors necessitating fractions are excluded, and in factoring, only integral factors are called for. But even all the integral factors are not required. First, every quantity is divisible by itself and unity, as the factors of a may be $1 \times a$; and this combination of factors is in algebra of very great use. But, as this is self-evident, these factors may be taken for granted, and need not be expressed. But this is not all. The factors of ab may be a and b, or they may be $-a$ and $-b$. The factors of $5ab$ may be (writing on board) $5 \times a \times b$, or $5 \times -a \times -b$, or $-5 \times -a \times b$, or $-5 \times a \times -b$. But we do not expect, in factoring, all these combinations. We want one combination of integral factors, with a preference for positive quantities.

We do not need to spend much time in factoring monomials; for, unless they have a large numerical factor, as $377xy$, they show what their factors are, especially of the literal part, as simply as it is possible to show them. A few examples will suffice.

Factor $2ax$. $9a^2b$; $9a^3y$; $15a^2b$; $5ab^2x$.

T. Factor $3x - 3y$.

NOTE 1. Show the pupils how to express this, and how to read it, namely, three times the quantity $x - y$, not "three times the parenthesis $x - y$," as some teachers do.

DRILL.

Factor, $5ab + 5ac$; $14ax - 7ay$; $15a^2b^2 - 30a^2b^2$; $5ax + 10ax^2 - 15ax^3$; $3a - 9a^2 + 6a^3$; etc.

NOTE 2. Next show the pupils that the factors may both be binomials, as $(a + 6)(a + 5)$; read "the quantity $a + 6$ multiplied by the quantity $a + 5$." Have them multiply these together, and show them the difference between the above and $a + 6(a + 5)$.

DRILL.

Combine, that is, multiply together and remove parentheses:

$(a + b)(a - b)$; $(x^2 + 6)(x^2 - 6)$; $(x - 5)(x + 5)$; $(2x - 3a)(2x - 3a)$. Continue this till the pupils see the relation between the terms of the product and those of the factors; then reverse the work and factor:

$x^2 - 4$; $a^2 - 9b^2$; $a^4 - b^2$; $25x^2 - 4y^2$, etc.

NOTE 3. Lastly, teach how to add and subtract quantities in parenthesis, as $x(x + 4)$ and $5(x + 4) = (x + 5)(x + 4) = x^2 + 9x + 20$; $x(x + a)$ minus $b(x + a) = (x - b)(x + a)$, etc.

PROBLEM. A and B had the same number of horses; A bought 4 more, and B sold 2; then twice the number A had equals six times the number B had; how many had each at first?

LESSON XVII.

The Parenthesis continued.

TEACHER. Factor:

$3\,a^2x - 3\,b^2x$ (three factors); $abx^2 - ab^3$; $5\,ax^2y^2 - 5\,a$; $7\,x - 28\,x^2 + 28\,x^3$; etc.

T. We have learned something about the use of the parenthesis, and now we must go a step farther.

It is sometimes used to enclose two or more terms of a polynomial to indicate that they are to be taken together as a whole, or *one term*, before being added to or subtracted from the rest. For example (putting on the board $a + (b - c)$): this, which should be read, a plus the quantity (not "parenthesis" or "expression," as some teachers read it) $b - c$ means that a quantity b greater and c less than zero, must be joined together before they are subtracted from a. To illustrate, let $a = 5$, $b = 4$, $c = 3$, and $d = 2$, and take the quantities $a + b - c + d$, and $a + (b - c + d)$.

The difference may be illustrated thus (see accompanying work): —

$$\begin{array}{rrr} 5 & & 5 \\ 4 & 4 & \\ -3 & -3 & \\ \underline{2} & 2 = & \underline{3} \\ 8 & & 8 \end{array}$$

The question, scholars, is this: What effect does placing a parenthesis around terms, or removing it, have on the signs of the terms enclosed? Remember that doing so ought not to change the total value of the quantity. In this case, you see, it makes no difference whether we have a parenthesis around the last three terms or not. Let us prove it another way; let us add the quantity in the parenthesis to a as the sign says it must be, just as we have heretofore added quantities together, thus: —

$$\begin{array}{r} b - c + d \\ \hline a + b - c + d \end{array}$$

You see the signs are exactly the same as when the terms were enclosed in a parenthesis.

NOTE. Take $2a + (b - 2c + 3d - 4)$, and go through the same steps.

T. It would seem, then, scholars, as if placing or removing a parenthesis were a very simple matter; but please observe that, in all the examples we have tested, the positive sign precedes the parenthesis. Shall we assume that it would be the same if that sign were negative, or had we better test it?

PUPILS. We had better test it.

T. So I think. Take $2a - (2b - c + d)$ and compute its value. Then remove the parenthesis and compute it. (The pupils do this with results as shown in (1) and (2).) There is evidently something wrong, scholars, and as the first is just what the example calls for, the error must be in the second.

$$(1)\quad \begin{array}{rr} & 10 \\ 8 & \\ -3 & \\ \hline 2 = & 7 \\ \hline & 3 \end{array} \qquad (2)\quad \begin{array}{r} 10 \\ -8 \\ 3 \\ 2 \\ \hline 1 \end{array}$$

Let us take the quantity in parenthesis, and subtract it from $2a$ as we have hitherto subtracted; thus:—

$$\begin{array}{r} 2a \qquad\qquad \\ 2b - c + d \\ \hline 2a - 2b + c - d \end{array}$$

(The pupils study this till they see that every term within the parenthesis, including $2b$, has changed its sign; and then they test it as in (3), getting the result as in (1).)

$$(3)\quad \begin{array}{r} 10 \\ -8 \\ 3 \\ -2 \\ \hline 3 \end{array}$$

NOTE 1. Take one more example in the same way and then reverse the process; i.e., enclose terms in parentheses, and repeat illustrations till the pupils see for themselves the law of signs.

NOTE 2. The theory of the parenthesis, which teachers may find it valuable to present to their pupils, is in accord with the theory of algebraic quantities set forth in Lesson VIII., and, briefly stated, is this: —

A polynomial is built up from a primal zero, of elements some or all of which may be either greater or less than zero. A parenthesis introduces a zero of its own which, with the terms connected with it, must either be added to, or subtracted from, the main zero. Let these zeros be represented by $0'$ and $0''$, then $a - (b - c + d)$ would be fully expressed thus: $0' + a - (0'' + b - c + d)$. Removing the parenthesis eliminates $0''$, and the signs referring to it are made to refer to $0'$. Hence it is evident that all quantities that were to be added to $0''$ before it was subtracted from $0'$, must be subtracted from $0'$; and all that were to be subtracted from $0''$, before it was taken from $0'$, must be added to $0'$.

So, too, in placing a parenthesis with, say, a negative sign before it, we introduce a $0''$ which is to be subtracted from $0'$; now all terms enclosed with $0''$, that originally were to be subtracted from $0'$, must be added to $0''$, and all quantities originally to be added to $0'$ must be subtracted from $0''$ before it is subtracted from $0'$.

The teacher can best judge whether or not to make use of this theory with young classes. The chief object is to cultivate facility in doing based on a clear understanding of the why.

T. Remove parentheses. (Rewrite in full.)

$$x^2 + (3x - 5)$$
$$x^2 - (3x - 5)$$
$$x^2 - (3ax + 2ay + 6)$$
$$3x^2 + 6x - (x^2 + 4x - 5)$$
$$a^4 - 5(a^3 + a^2 - a - 1), \text{ etc.}$$

Enclose the last two terms in parentheses.

$$ax^2 - bx + c$$
$$a^3 + 3\,ax + 2\,a^2$$

Enclose the last three terms in parentheses.

$a^4 - 4\,a^2b + 5\,ab - b^3$
$a^3 - 3\,a^2b + 3\,ab^2 - b^3$
$x^3 - 8\,x^2 + 8\,x - 8$ (factor quantity in par.), etc.

NOTE. After sufficient drill in placing and removing parentheses, give the pupils examples like the following: —

$(a+b)\,(a-b)$; $(a+3)\,(a-3)$; $(2\,x-4)\,(2\,x+4)$; $(3\,ab-2\,a)$ $(3\,ab+2\,a)$.

Factor:

$a^2 - c^2$; $4\,x^2 - y^2$; $4\,x^2 - 9\,y^2$; $4\,a^2b^2 - x^2y^2$; $5\,m^2n^2 - 20\,x^2$ (divide by 5); $25\,x^4y^4 - 9\,a^2$; $1 - 64\,a^2$; $81\,x^2y^4 - 1$; $49 - a^4x^2$; $a^6x^2 - 100$; $\frac{1}{a^2} - 1$; $1 - \frac{4}{x^2y^2}$; etc.

LESSON XVIII.

Multiplication at Sight — Factoring — The Use of Two Letters to Express the Unknown Quantities of a Problem.

NOTE. In this and the following lessons the work will be only suggested as below. "Etc." at the end of a series of examples means, give the pupils enough drill to make the point clear. The teacher should be prepared to make simple problems as occasion requires.

I. Multiply: $a + b$ by $a + b$; $x + 5$ by $x + 5$, etc.

II. Factor: $x^2 + 2\,xy + y^2$; $x^2 + 4\,x + 4$; $x^2 + 8\,a\,x + 16\,a^2$, etc.

III. Multiply: $a - b$ by $a - b$; $x - 3$ by $x - 3$, etc.

IV. Factor: $x^2 - 2xy + y^2$; $x^2 - 8x + 16$, etc.

V. Multiply: $a + b$ by $a + c$; $x + 5$ by $x + 3$, etc.

VI. Factor: $x^2 + 3x + 2$; $x^2 + 4x + 3$; $x^2 + 9x + 14$, etc.

VII. Multiply: $a - b$ by $a - c$; $x - 2$ by $x - 3$, etc.

VIII. Factor: $x^2 - 5x + 6$; $x^2 - 7x + 6$, etc.

IX. Find two quantities by inspection,

Whose sum is 7, and product 6.
Whose sum is 10, and product 20.
Whose sum is − 8, and product 12.
Whose sum is − 2, and product − 15, etc.

Problems. $x + y = 25$; $x - y = 7$; $3x + 2y = 23$; $2x + y = 15$.

$$x + y = 25; \qquad 3x + 2y = 23.$$
$$x - y = 7; \qquad 2x + y = 15.$$

LESSON XIX.

Factoring arranged by Cases.

Note. Factoring is of such importance that this lesson is devoted to a further study of it as arranged under separate cases. The first six cases will be mainly a review; the seventh introduces a new combination, which should be taught as the others, first by synthesis.

I. Multiply: $a + b$ by $a - c$; $x + 5$ by $x - 4$; $x - 3$ by $x + 5$, etc.

II. Factor: $x^2 + 2x - 15$; $x^2 + x - 6$, etc.

Problem.

$$\frac{x}{2} + \frac{y}{5} = 3;$$

$$\frac{3x}{4} - \frac{y}{5} = 2.$$

LESSON XX.

Greatest Common Divisor.

Note. In finding the greatest common divisor, factor by inspection. The process of finding it by dividing one quantity by the other need not be used in this book except in miscellaneous examples at the end.

Problems.

(1). $\frac{1}{5}+\frac{1}{4}=\frac{1}{x}$.

(2). A can do a piece of work in 20 hours, and B in 30; how long will it take them both to do it?

LESSON XXI.

Factoring continued — Greatest Common Divisor continued.

I. Divide a^3+b^3 by $a+b$; a^5+b^5 by $a+b$; $x^3+27 = x^3+3^3$ by $x+3$, etc.

II. Factor: x^3+y^3; x^5+y^5; x^3+64; x^5+64, etc.

III. Divide x^8-b^8 by $a-b$; a^5-b^5 by $a-b$; x^8-8 by $x-2$, etc.

IV. Divide: $(a+b)^8$ by $(a+b)^2$; $(x-6)^5$ by $(x-6)^3$, etc.

LESSON XXII.

Fractions — Axioms.

I. Fractions may have monomials or polynomials for numerator or denominator, or both.

II. Reduce to lowest terms by dividing both numerator and denominator by the greatest common divisor.

III. Give the axioms in full:

1. Equals added to equals, etc.
2. Equals subtracted from equals, etc.
3. Equals multiplied by equals, etc.
4. Equals divided by equals, etc.

PROBLEM. A cistern holds 60 gallons; A and B working together 10 minutes filled it; at another time A working 12 minutes and B working 4, filled it; how many gallons did each pump in each minute?

Ans. A 10, and B -4.

Interpret the negative result.

LESSON XXIII.

Least Common Multiple — Factoring continued, Case X.

I. Least Common Multiple.

1. Give a multiple of a; of b; of $3x$, etc.

2. A common multiple of a and b; of $2a$, $3b$, and c; of $3a$, $3x$, and $3y$; $a+b$, and $a-b$; of $x+2$ and $x-2$, etc.

3. The Least Common Multiple of a^2 and ab^2; of $a^2 - b^2$ and $a + b$; of $x^2 - 3x + 2$, and $x^2 + 3x - 4$, etc.

II. Combine: $(a + b)(a - c)$; $(x + 5)(x - 3)$; $(x - 4)(x + 1)$, etc.

III. Factor: $a^2 - 3a - 4$; $x^2 + 2x - 15$; $x^2 - 2x - 15$; $x^2 + 3x - 28$; $x^2 - 3x - 28$, etc.

IV. Add: $4(x + 2)$, $3(x + 2)$, and $5(x + 2)$; $7(a + b)$, $-3(a + b)$, and $2(a + b)$; $3(a - 2)$, $-6(a - 2)$, $4(a - 2)$, and $a - 2$, etc.

LESSON XXIV.

Use of the L. C. M. — Addition and Subtraction of Fractions.

I. Change to a Common Denominator: $\frac{a}{3}$ and $\frac{a}{5}$; $\frac{a}{b}$ and $\frac{a}{c}$; $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$; $\frac{x-2}{x-3}$ and $\frac{x-4}{x+3}$, etc.

II. Add: $\frac{x+4}{x+5}$ and $\frac{x-3}{x-5}$; $\frac{x+7}{x-4}$ and $\frac{x+3}{x+5}$, etc.

III. Subtract: $\frac{a+6}{a-6}$ from $\frac{a+7}{a-7}$, etc.

IV. Unite as indicated by signs: $\frac{x-5}{x+4} - \frac{x-3}{x+3}$; $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)}$; $\frac{x-2}{x+1} - \frac{x+2}{x-1} - \frac{x-3}{x^2-1}$, etc.

Note. Observe that in uniting fractions by changing them to a common denominator and combining their numerators, the signs of the terms of the several numerators follow the law for removing parentheses; see Lesson XVII.

LESSON XXV.

Mixed Numbers to Fractions — Fractions to Mixed Numbers.

1. Change to a fraction: $1+\frac{a}{b}$; $5-\frac{4}{x}$; $a-\frac{ax}{x+3}$, etc.

2. Reduce to mixed number;

$$\frac{a^2+b^2}{a+b};\ \frac{a^3+b^3}{a-b};\ \frac{a^5-b^5}{a+b};\ \frac{x^2-4x+7}{x-5},\ \text{etc.}$$

Problem. A farmer had c bushels of apples; he lost a third of them, and then a fourth of the remainder; how many had he left? *Ans.* $\frac{c}{2}$.

LESSON XXVI.

Multiplication and Division of Fractions by Whole Numbers.

Note. In these operations factor whenever possible.

Multiply $\frac{x+3}{x^2-3x+2}$ by $x^2+6x-16$; etc.

Divide $\frac{x^2-5x+4}{x-7}$ by x^2+6x-7; etc.

Problem. A stream flows at the rate of one mile an hour. A boat's crew rowed down the stream for 2 hours, and were 2½ hours in rowing back. How fast can the crew row in *still water?*

LESSON XXVII.

Multiplying Fractions or Mixed Numbers by Fractions or Mixed Numbers.

I. Multiply:

$$a+\frac{b^2}{a} \text{ by } a-\frac{b^2}{a};\ 1-\frac{6}{x+4} \text{ by } 1+\frac{8}{x-5},\ \text{etc.}$$

LESSON XXVIII.

Equations of the First Degree.

TEACHER. Scholars, while working on the problems in this and the three following lessons, there are four axioms which it will be well for you to keep in mind. But first, in how many ways have you observed that a known quantity can be connected with x?

PUPILS. Four: by addition, by subtraction, by multiplication, and by division. (Illustrations are put on board.)

T. In an equation, how would you eliminate the unknown quantity in each case?

P. If connected by addition, subtract it from both members, etc.

T. Give the four axioms on which these operations are based. (The pupils do so.)

T. There are two other cases, but they will come up later.

NOTE. For other suggestions see note to Lesson XXVIII. of Student's Manual.

LESSON XXIX., XXX., AND XXXI.

NOTE. See note to Lesson XXVIII. of Student's Manual.

LESSON XXXII.

Involution continued — Application of the Binomial Theorem.

I. Expand:

1. $(a \pm b)^2$; $(a \pm b)^3$; $(a \pm b)^4$, etc.
2. $(x + 2)^3$; $(2x - 3)^5$, etc.

Note. Let these be first written thus: —

$$(2x - 3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$$
$$= 8x^3 - 36x^2y + \text{etc.}$$

LESSON XXXIII.

Evolution, Square Root — Pure Equations of the Second Degree.

I. Formula, $a^2 + 2ab + b^2 = a^2 + (2a + b)b$.

1. $\sqrt{x^2 + 18x + 81}$, etc.
2. $\sqrt{4225}$; $\sqrt{2704}$; $\sqrt{6}$ (two dec.), etc.

Teacher. Scholars, in a previous lesson you said there were four ways in which a known quantity could be connected with x. (Review them.) I told you then that there were two other cases; can you tell me what they are?

If the scholars cannot think of any other ways the teacher can suggest $x^2 = 9$, and $\sqrt{x} = 7$.

Note. The pupils are then led to see that by analogy with the other cases, the exponent of a power should be eliminated by extracting a root, and the index of a root by raising to a power.

II. Solve: $x^2 = 9$; $x^2 = 16$; $x^3 = 27$; $\sqrt{x} = 5$; $\sqrt[3]{x} = 3$, etc.

NOTE. If the pupils want to know why $x^2 = 9$ does not give $\pm x = \pm 3$, rather than $x = \pm 3$, tell them you want to find the value of x, not $-x$.

PROBLEM. $x^2 + y^2 = 13.$
$x^2 - y^2 = 5.$

LESSON XXXIV.

Evolution, Cube Root.

Formula: $a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$

LESSON XXXV.

Roots of Equations — Factoring, Case XI.

I. A root of an equation is a value of x that will satisfy the equation.

II. What value of x will satisfy the following equations of the first degree?

$$x - 2 = 0;\ x + 3 = 0, \text{ etc.}$$

How many roots has an equation of the first degree?

III. Factor the first member of the following equations of the second degree, and find how many values of x will satisfy each, and how many roots each has.

$$x^2 - 7x + 10 = 0; \qquad x^2 + 9x + 20 = 0;$$
$$x^2 - 3x - 4 = 0; \qquad x^2 + 5x - 24 = 0, \text{ etc.}$$

IV. Multiply:

1. $ax + b$ by $ax + c$; $3x + 5$ by $3x + 7$; $2x - 8$ by $2x + 6$, etc.

V. Factor: $4x^2 + 20x + 21$; $9x^2 - 21x + 10$; $4x^2 - 2x - 12$; etc.

LESSON XXXVI.

Affected Quadratic Equations solved by completing the Square.

I. $\sqrt{x^2 + 8x + 16}$, etc.

II. Add a quantity to the following quantities that will make them perfect squares, and extract the square root: $x^2 + 10x$; $x^2 - 6x$; $x^2 - 9x$, etc.

III. Add to both members of the following equations what will make the first member a perfect square, and find the roots: —

$x^2 + 6x = 55$; $x^2 - 8x = 33$; $x^2 + 11x = 12$; $4x^2 - 16x = -12$, etc.

LESSON XXXVII.

Problems producing Affected Quadratic Equations.

NOTE. Assist as needed.

LESSON XXXVIII.

Arithmetical Examples solved by Algebraic Formulas.

NOTE. Make clear the general formulas.

LESSON XXXIX.

Factoring, Case XII. — Permanences and Changes in Signs.

I. When in an algebraic quantity two successive terms have the same sign, it is called a permanence; when two successive terms have different signs, it is called a change. For example, in $3x^2 + 4x + 1$, the first term being positive, there are two permanences; in $5x^2 + x - 20$, the sign of x is a permanence, the sign of -20 a change; in $3x^2 - 5x - 5$, the sign of $-5x$ is a change, that of -5 a permanence; in $2x^2 - 7x + 6$, the signs of both $-7x$ and 6 are changes. In the following examples observe the permanences and changes in the signs of the product, (1), when the signs connecting the terms of the binomial factors are both positive; (2), when one is positive and the other negative; and (3), when both are negative.

Derive a law for determining the signs connecting the terms of the binomial factors of a quantity.

II. General Formulas:

$$(ax + b)(x + c) = ax^2 + (ac + b)x + bc.$$

$$(ax + b)(cx + d) = acx + (ad + bc)x + bd.$$

Combine $(5x + 7)(x - 2)$, etc.

Factor $7x^2 + 6x - 16$, etc.

Combine $(2x + 3)(4x - 5)$, etc.

Factor $10x^2 - x - 3$, etc.

MacDonald's Mathematical Series.

PRIMARY ALGEBRA.

STUDENT'S MANUAL.

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PRIMARY ALGEBRA.

To the scholars. — Strive to work out the problems in these lessons without help.

LESSON I.

Problems to be Solved by the Use of x.

1. Four times, seven times, and three times a number is 294; what is the number?

2. Three men, A, B, and C, divide $288 in such a way that B has twice as much, and C five times as much as A; how much has each?

3. A post 28 feet long stands six times as much above ground as in the ground; how deep is it in the ground? How long above ground?

4. A general distributed 2400 men in three forts so that fort No. 2 had twice as many as fort No. 1, and fort No. 3 had as many as in both the others; how many did he put in each fort?

5. Divide the number 534 into two parts, so that one part will be five times the other.

6. A string 125 inches long is cut so that one piece is four times as long as the other; how long is each?

7. A man bought a horse, a carriage, and a harness for $225. He gave four times as much for the carriage as for the harness, and as much for the horse as he did for both the carriage and the harness; how much did he give for each?

8. $3x + 4x + 6x = 403$, to find value of x.

9. $5x + 2x = 154$, to find value of x.

10. I bought a number of apples for two cents each, and the same number of oranges for three cents each, paying 55 cents for the whole; how many did I buy?

LESSON II.

Addition and Subtraction of Positive Quantities — Problems.

1. Three men, A, B, and C, had $27m$ acres of land, of which B had three times as much as as A, and C five times as much as A; how many had each?

2. A boy had some oranges, three times as many apples, and as many pears as apples and oranges together; the whole number was 240. How many had he of each?

3. Find the sum of the following quantities, and also the difference by subtracting the second from the first, arranging them thus:—

NO.	QUANTITIES.		SUM.	DIFFERENCE.
1	$4m$	$2m$	$6m$	$2m$
2	$25c$	$18c$	$43c$	$7c$

1. $7a$ and $4a$.
2. $9b$ and $3b$.
3. $12m$ and $8m$.
4. $8a$ and $3m$.
5. $6c$ and $5b$.
6. $25b$ and $22a$.
7. $4c + 8$ and $3c + 6$.
8. $9d + 12$ and $6d + 12$.
9. $5a + 7$ and $5a + 4$.
10. $5a + 6c$ and $3a + 3c$.
11. $7d + 8c$ and $4d + 8c$.
12. $12a + 9m$ and $12a + 8m$.

Sum and difference of

4. $13b + 7c + 18$ and $13b + 6c + 15$.

5. $5m + 6n + 7s - 8$ and $4m + 6n + 7s$.

6. $25a + 13b + 8c$ and $25a + 13b + 8c$.

7. Find the value of x in each of the following equations, if $a = 5$, $b = 4$, and $c = 10$.

1. $x = 6a$.
2. $x = 3a - 2b$.
3. $x = 4b - c$.
4. $x = 5a - 4b + c$.
5. $x = 3c - 5a - 4b$.
6. $x = 10a - 4b - 20$.
7. $x = 50 - 3c - 2b$.
8. $x = 10b - 4a - 20$.

8. Two railroad trains 165 miles apart are approaching each other, one at the rate of 30 miles, and the other 25 miles per hour; how long before they will meet?

9. Two ships are sailing towards each other at the rates of 8 and 12 miles per hour respectively. If the distance between them is 130 miles, how long before they will meet? How far will each have sailed?

10. In an orchard of 162 trees there are five times as many apple-trees as peach-trees, and half as many pear-trees as of both the other kinds together; how many are there of each?

11. To $5a + 6b + 2c$ add $a + 3b + 3c$; then subtract $4a + 8b + c$; then add $7a + 6b + 3c$; then subtract $9a + 7b + 6c$. What is the remainder?

12. Take $4a$; add $3a + 7b$; subtract $5b$; add $3a + 8b + 2c$; add $2a + 8b + 5c$; subtract $9b + 6c$; subtract $6a + 5b$; add $2a + 4b + 7c$; divide by 8. *Ans.?*

13. Form the following quantities:

1. One 6 larger than x; one 6 times x.
2. One 8 larger than a; one 8 times a.
3. One 5 smaller than x; one 5 times x.
4. One 7 smaller than b; one 7 times b.
5. One 10 larger than x; one 10 times x.

14. Form quantities:

1. a larger than x; a smaller than x.
2. b larger than m; b smaller than m.
3. $3b$ larger than y; $3b$ smaller than y.
4. $3c$ larger than $2b$; $3c$ smaller than $2b$.

15. Form numbers:

1. $a + b$ larger than $b + c$.
2. $m + n$ larger than $3m$.
3. $a + c$ smaller than $2a + c$.
4. $2a + 3b + 6$ smaller than $2a + 4b + 12$.

LESSON III.

Multiplication and Division — Exponents.

1. Form quantities:

1. a larger than y; a smaller than y.
2. $2b$ larger than $3a$; $2b$ smaller than $3a$.
3. $4m$ larger than $4x$; $4m$ smaller than $4x$.

2. Answers to the following:

1. a^4 multiplied by a^3; a^4 divided by a^3.

2. a^5 multiplied by a^2; a^5 divided by a^2.
3. b^5 multiplied by b^3; b^5 divided by b^3.
4. $75 c^2$ multiplied by $5 c$; $75 c^2$ divided by $5 c$.
5. $125 s^4$ multiplied by $25 s^3$; $125 s^4$ divided by $25 s^3$.
6. $6 m^2$ multiplied by $3 m^2$; $6 m^2$ divided by $3 m^2$.
7. $12 b^5$ multiplied by $6 b^4$; $12 b^5$ divided by $6 b^4$.

3. Five times a certain number and eight times the same number is 156; what is the number?

4. John has a number of cents, Henry has three times as many, Willie has as many as John and Henry together, and Fred has three-fourths as many as the other three; altogether they have $1.68. How many has each?

5. Add $5 a^2$, $7 a^2$, $3 a^2$, $4 a^2$, and $9 a^2$ together, and multiply the sum by $3 a^3$.

6. Add $6 x^3$, $9 x^3$, $2 x^3$, x^3, $3 x^3$, and $7 x^3$ together, and divide the sum by $14 x^2$.

7. Add together $5 b$, $7 b$, $8 b$, $9 b$, $6 b$, and $5 b$, and divide the sum by $20 b^2$.

8. Add together $4 c$, $8 c$, $5 c$, $6 c$, and $2 c$, and multiply the sum by $5 c^3$.

9. Lucy has 15 cents more than Mary, and both together have 47 cents; how many have each?

10. In an election one candidate received 79 votes more than the other out of a total vote of 1,637; how many did each receive?

11. The difference between two numbers is 23, and their sum is 71; what are the numbers?

12. In an election A received a certain number of votes, B received 25 more than twice as many, and C 8 less than three times as many. The total vote was 3,227; how many did each receive?

LESSON IV.

Multiplication and Division continued—Problems.

1. A man's age is five times his son's age, the sum of their ages is 36; what is the age of each?

2. If three times and five times a number be added to the number, the sum will be 279 m; what is the number?

3. Form the following quantities:

1. 4 larger than a; 4 times a.
2. a larger than x; a times x.
3. ab larger than y; ab times y.
4. d smaller than x; d times x.
5. a^2d smaller than a^2; a^2d times a^2.

4. Do the following examples:

1. a multiplied by b; a divided by b.
2. a^2x multiplied by a^2b; a^2x divided by a^2b.
3. b^2 multiplied by b^2; b^2 divided by b^2.
4. a^2bx multiplied by abx; a^2bx divided by abx.
5. a^4x^2 multiplied by a^5x; a^4x^2 divided by a^5x.
6. m^2 multiplied by m^3; m^2 divided by m^3.
7. $4\,x^3$ multiplied by $2\,x^2$; $4\,x^3$ divided by $2\,x^2$.
8. $16\,a^2x$ multiplied by $8\,a^3$; $16\,a^2x$ divided by $8\,a^3$.
9. $25\,b^3$ multiplied by $5\,b^3d^2$; $25\,b^3$ divided by $5\,b^3d^2$.

5. Multiply $a^2x + ax + x$ by a^3x^2.

6. Divide $15\,a^3b + 25\,a^2b^2 + 15\,ab^3$ by ab.

7. Divide $5\,a + 10\,b$ by ab.

8. Multiply $3\,ab + 4\,a^2b^2 + 5\,a^3b^3$ by $2\,a^2b^2c$.

9. Multiply $9\,ax + 7\,a^2x + 6\,a^3x^2$ by $4\,ax^2$.

10. Divide $4m^2n + 6mn^2 + 8mn$ by $2mn$.

11. Multiply $3a + 4b + 5c$ by abc.

12. $3x + 25 = 109$, to find the value of x.

13. $5x - 60 = 80$, to find the value of x.

14. $6x + 12 = 78$, to find the value of x.

LESSON V.

Review — Negative Quantities.

Add the following:

1.	**2.**	**3.**	**4.**
$3x$	$5a^2$	$7bc^2$	$12x^2y$
$7x$	$6a^2$	$9bc^2$	$15x^2y$
$4x$	$7a^2$	$18bc^2$	$18x^2y$
$5x$	$8a^2$	$22bc^2$	$25x^2y$
$6x$	$4a^2$	$36bc^2$	$9x^2y$
$7x$	$3a^2$	$8bc^2$	$12x^2y$
$2x$	$9a^2$	$4bc^2$	$42x^2y$

5. From $9x^2y + 8ab + 19c^2$ subtract $8x^2y + 8ab + 12c^2$.

6. Add:

1. x to a^2.
2. b to c.
3. m to n.
4. $3b$ to $7a$.
5. $3a$ to $4b$.
6. $2c$ to d.

7. Subtract:

1. 9 from x.
2. $3b$ from a.
3. $2c$ from m.
4. $5d$ from 5.
5. $3x$ from $2d$.
6. 7 from $4x$.

8. Maine and Vermont together have an area of 42,605 square miles, and Maine has 23,475 square miles more than Vermont; what is the area of each?

9. Massachusetts and Connecticut have together an area of 13,305 sq. miles, and Massachusetts has 3,325 sq. miles more than Connecticut; what is the area of each?

10. New York, New Jersey, and Pennsylvania have together an area of 102,200 sq. miles. New York has 3,955 sq. miles more than Pennsylvania, and New Jersey has 37,400 sq. miles less; what is the area of each?

11. $x + 2a = 5a$, to find value of x in terms of a.

12. $3x + 6a = 9b$, to find value of x in terms of a and b.

13. $3x + 4a = 8a + x$, to find value of x in terms of a.

14. $5x - 7a = 2a + 9b + 2x$, to find value of x.

15. From the sum of $2m + 3n + 2x$, $7m + 3n + 5x$, $8m + 4x$, $3n + x$, and $5m + 2x$, subtract the sum of $3m + 8x$, $6n + 2x$, and $5m + 3n$.

16. Farmer A has five times as many bushels of wheat as farmer B, and farmer C has 10 bushels more than both of them. Altogether they have 82 bushels; how many has each?

17. Read these quantities:

$-7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,$

$-7a,\ -6a,\ -5a,\ -4a,\ -3a,\ -2a,\ -a,\ 0,\ a,\ 2a,\ 3a,\ 4a,\ 5a,\ 6a,$

$-7b,\ -6b,\ -5b,\ -4b,\ -3b,\ -2b,\ -b,\ 0,\ b,\ 2b,\ 3b,\ 4b,\ 5b,\ 6b,$

$-7x,\ -6x,\ -5x,\ -4x,\ -3x,\ -2x,\ -x,\ 0,\ x,\ 2x,\ 3x,\ 4x,\ 5x,\ 6x.$

18. In the following pairs of quantities state which is the larger and how much; as, for example:

$5a$ is $7a$ larger than $-2a$.

6 or 2. $5b$ or $7b$. $7a$ or $3a$.

6 or -2. $5b$ or $-7b$. $-7a$ or $-3a$.

-6 or -2.
-5 or -2.
8 or 0.
-8 or 0.
0 or $-6a$.
$3b$ or $-3b$.

$9x$ or $-x$.
$-9x$ or x.
$-12x$ or $-3x$.
$-2x$ or $7x$.
$9x$ or $8x$.
$-9x$ or $-8x$.

$-2a$ or $-9a$.
$2a$ or $9a$.
$7a$ or $4a$.
$-7a$ or $-4a$.
6 or -3.
-6 or -3.

19. In the following give the smaller in the same way:

$7a$ or 0.
$-7a$ or 0.
-8 or 0.

$9a$ or $-9a$.
0 or $-4x$.
0 or $-2b$.

$6b$ or $-7b$.
$-8x$ or $-9x$.
$-8y$ or $8y$.

20. Write a quantity:[1]

6 larger than -6.
2 larger than -7.
7 larger than -7.
3 larger than -2.
4 larger than -2.
3 larger than -6.
$4b$ larger than $-4b$.
$3a$ larger than $-7a$.

$2a$ larger than $-2a$.
$6b$ larger than $-12a$.
$9b$ larger than 0.
$5x$ larger than $-3x$.
$8x$ larger than $-12x$.
$6b$ smaller than $-8b$.
$7a$ smaller than 0.
$7b$ smaller than $3b$.

Add:

21.
$$\begin{array}{r} 5a \\ -2a \\ -3a \\ -4a \\ 6a \\ -2a \\ 3a \end{array}$$

22.
$$\begin{array}{r} 5x \\ -3x \\ -5x \\ 8x \\ -4x \\ 3x \\ -4x \end{array}$$

23.
$$\begin{array}{r} 8b \\ 2b \\ -9b \\ -3b \\ -7b \\ -2b \\ 4b \end{array}$$

24.
$$\begin{array}{r} 12c \\ -12c \\ -7c \\ 3c \\ 4c \\ 8c \\ -7c \end{array}$$

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[1] Or give them orally.

LESSON VI.

Addition (Negative Quantities) — Problems.

Add the following:

1.
$$\begin{array}{r} 9 \\ -7 \\ 2 \\ 8 \\ -4 \\ -5 \\ 2 \\ 7 \\ 6 \\ 3 \\ -2 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 4a \\ -5a \\ 7a \\ -2a \\ 3a \\ -6a \\ -a \\ 7a \\ 8a \\ -2a \\ 3a \\ \hline \end{array}$$

3.
$$\begin{array}{r} 2x \\ -3x \\ -4x \\ 5x \\ 6x \\ -7x \\ -5x \\ -6x \\ 3x \\ 7x \\ -2x \\ \hline \end{array}$$

4.
$$\begin{array}{r} -12b \\ 210b \\ 375b \\ -621b \\ 35b \\ 81b \\ -273b \\ 836b \\ -239b \\ 47b \\ -99b \\ \hline \end{array}$$

NOTE. Do not add the positive and negative quantities separately, but in order as they occur. This is a good drill to cultivate watchfulness as to signs.

5.
$$\begin{array}{r} 5ab \\ -7ab \\ -31ab \\ 42ab \\ 54ab \\ 22ab \\ \hline \end{array}$$

6.
$$\begin{array}{r} -9x^2 \\ 12x^2 \\ -13x^2 \\ 24x^2 \\ -54x^2 \\ 39x^2 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 91ax^2 \\ 42ax^2 \\ -36ax^2 \\ -2ax^2 \\ 16ax^2 \\ -5ax^2 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 5a^2 \\ -17a^2 \\ -29a^2 \\ 37a^2 \\ -4a^2 \\ -9a^2 \\ 3a^2 \\ -3a^2 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 14ax \\ -36ax \\ 97ax \\ -69ax \\ -24ax \\ 32ax \\ 91ax \\ -27ax \\ \hline \end{array}$$

10.
$$\begin{array}{r} 6y \\ 7y \\ 8y \\ 4y \\ 3y \\ 8y \\ 9y \\ 7y \\ \hline \end{array}$$

11. $\frac{x}{2} + 9 = 25$, to find value of x.

12. $\frac{2x}{3} + \frac{1}{3} = 27$, to find value of x.

13. A farmer, when asked how many cows he had, replied that if he had as many more, and half as many more, he would have 75; how many had he?

14. A boy told his companions that if he had two-thirds as many cents more than he then had, plus 4 cents, he would have 49 cents; how many had he?

15. In a certain school of 84 scholars there are three-fourths as many boys as girls; how many are there of each?

16. A farmer has a number of cows in one herd, three-fourths as many in a second, and 4 more than half as many in a third; in all there are 76 cows. How many in each herd?

17. If to a certain number, itself and one-fifth of itself and 3 be added, it will give 80; what is the number?

18. Meeting a man driving a flock of geese to market, I said, "Good-morrow, master, with your hundred geese." Said he, "I have not a hundred; but if I had as many more, half as many more, and 2½ geese, I would have a hundred." How many had he?

19. One-third of a number plus one-fourth of it is 7; what is the number?

NOTE. It may take two lessons to do all these examples, in which case spend a part of each lesson on drill in addition.

LESSON VII.

Addition continued—Problems.

Add:

1.
$$\begin{array}{r} 3x \\ -5x \\ 7x \\ -6x \\ 4x \\ -3x \\ 5x \\ \hline \end{array}$$

2.
$$\begin{array}{r} 6a \\ -2a \\ -a \\ -7a \\ -5a \\ -3a \\ 19a \\ \hline \end{array}$$

3.
$$\begin{array}{r} 6 \\ -7 \\ -3 \\ -2 \\ -4 \\ 8 \\ -3 \\ \hline \end{array}$$

4.
$$\begin{array}{rr} 2x & +4 \\ 3x & -2 \\ -4x & +3 \\ -7x & -2 \\ 8x & -4 \\ 7x & -6 \\ \hline \end{array}$$

5.
$$\begin{array}{rrr} 7x^2y & -3ab^2 & +8d \\ -5x^2y & -2ab^2 & -5d \\ 3x^2y & +4ab^2 & +2d \\ -2x^2y & +ab^2 & -d \\ 3x^2y & -2ab^2 & +3d \\ -8x^2y & +5ab^2 & -2d \\ 3x^2y & -6ab^2 & -5d \\ \hline \end{array}$$

6.
$$\begin{array}{rr} 4x & +c \\ -3x & -c \\ 6x & +4c \\ 5x & -c \\ -7x & -6c \\ -4x & +2c \\ -x & +c \\ \hline \end{array}$$

Arrange and add the following:

7. $7x, 3a, -4b, -3x, 5a, -7b, -9c, 2x, -2a, 9c, 4b, -6x, 3a, 7b.$

8. $5a, -7, -2a, 3b, 5, 6a, -2.$

9. $3b-2a+6c,\ 2c-5b+2a,\ 3a-4c+3b,\ 7b-a+3c.$

10. $4a, -3c, 2b, -2a, 5c, -3b, -a, 2c.$

11. $8x, 3y, -4x, 3z, -4z, -7x, 3x, -3y, z.$

Find quantities:

12.
a larger than x.
b larger than $x-b$.
c smaller than $x+c$.

$x + a$ larger than $x - a$.
$a + 3$ larger than $a - 3$.
$b - 3$ smaller than $b - 3$.

13. In a library two-fifths of the books are histories, one-third scientific works, and the remaining 240 are poems; how many volumes in the library, and how many of each kind?

14. Three-fourths of a river is in one State, one-eighth in another, and 24 miles of it in another; how long is the river?

15. $\frac{4x}{3} + \frac{3x}{4} + 50 = 568 - x$.

16. $x - 40 = 40 - \frac{x}{3}$.

17. A number is as much larger than 25 as 25 is larger than two-thirds of the number; what is the number?

18. A farmer sold two-fifths of his farm to one man, one-third to another, and had forty acres left; how large was his farm at first?

19. Henry had 8 marbles more than Fred, and Willie had 5 more than both the others; altogether they had 61 marbles. How many had each?

20. Add $7xy$, $-4ay$, $-3xy$, $-3ax$, $4ay$, $-4xy$, and ax.

LESSON VIII.

Subtraction — Miscellaneous Problems.

1. From 9 take 7.

2. From 7 take 9.

3. From -9 take -7.

4. From -7 take -9.

5. From $9a$ take $5a$.

6. From $5a$ take $9a$.

7. From $-9a$ take $-5a$.

8. From $-5a$ take $-9a$.

9.[1] From $7a^2 - 6ab + 4ac + 7c^2$, take $5a^2 - 9ab + 6ac + 7c^2$.

10. From $13m^2 + 8mn + 24n^2 - 9$ take $6m^2 + 8mn - 24n^2 - 18$, and add to the remainder $-6m^2 - 48n^2 - 9$.

Note. Rule a table of six columns as follows: —

No.	Quantities.		Sum.	1st — 2d.	2d — 1st.
	1st.	2d.			
1	$6a$	$4a$	$10a$	$2a$	$-2a$
2	$-8x^2$	$-5x^2$	$-13x^2$	$-3x^2$	$3x^2$

11. Fill out as above from the following quantities: —

1. 12 and 6.
2. -12 and -6.
3. -12 and 6.
4. 12 and -6.
5. $8a^2$ and $5a^2$.
6. $-8a^2$ and $-5a^2$.
7. $13x^2y$ and $-7x^2y$.
8. $7ab$ and $-7ab$.
9. m and $-m$.
10. $-7d$ and $-12d$.
11. $24a^3$ and 7.
12. $36a^2$ and $12a$.
13. $5xy$ and $4c$.
14. $8b$ and $8c$.
15. $7ax$ and $7ax$.
16. $9by$ and $-9by$.
17. $3a^3$ and $4a^2$.
18. $8x^2$ and $-8x^2$.
19. 12 and 12.
20. -12 and -12.

---

[1] In subtraction *do not change*, or *suppose changed*, the sign of the subtrahend, and then add. Subtract by finding the difference.

12. Rule a similar table, and fill out from the following binomial quantities: —

1. $a+b$ and $a-b$.
2. $x+7$ and $x-7$.
3. $6b+3c$ and $7b-9c$.
4. $5ab+4x$ and $5ab-4x$.
5. $y+x$ and $y-x$.
6. $3d+9$ and $4d-6$.
7. $8+2x$ and $8-2x$.
8. $12+3a$ and $7-4a$.
9. $13+x$ and $13-x$.
10. $15-9y$ and $14-9y$.

13. $x+30=\frac{x}{2}+22$, to find value of x.

14. One-third of a man's property minus one-half of it equals 100 dollars; what is his property? How do you explain the answer? *Ans.* $— 600.

15. One day I deposited in a bank a certain sum of money, the next day I deposited the same sum plus 9 dollars; if I had deposited 43 dollars more, I would have deposited in all 30 dollars. How large was my deposit each day? Interpret the answer. *Ans.* $ — 11.

16. A certain number and two-thirds of the number equals 45; what is the number?

17. Two-fifths of a road is through forests, one-third through fields, and $2\frac{2}{3}$ miles along a river bank; how long is the road?

18. From 0 subtract each of the following quantities: $6m$; $-6m$; $-3ax$; $16a^8b$; $5ay$; $-8d$; $a-b$; $1-a$.

19. Write a quantity $5a$ larger than $-10a$; $7x$ larger than $-4x$; 16 larger than -16.

20. What must be added to a to make it b? to $-a$ to make it b?

LESSON IX.*

Miscellaneous Problems.

Add:

1.	**2.**	**3.**	**4.**
$7x$	$75ab$	$345a^2$	$8b$
$-4x$	$-33ab$	$227a^2$	$-11b$
$-5x$	$42ab$	$-364a^2$	$5b$
$3x$	$-16ab$	$843a^2$	$-9b$
$8x$	$-27ab$	$-279a^2$	$13b$
$-2x$	$-38ab$	$-184a^2$	$-6b$

Subtract each from the other:

5. $5a^2 - 6ab + 3b^2$, and $4a^2 - 2ab + 7b^2$.

6. $13a^3 - 39a^2b + 39ab^2 - 13b^3$, and $5a^3 - 15a^2b + 15ab^2 - 5b^3$.

7. Multiply: a^4 by ab^3; $2x^3y$ by xy^3; ab^3 by a^2bc; $5a^2bc$ by $5a^2bc$.

8. Multiply $5a^3 + 7a^2b + 5ab^2 + 2b^3$ by $3a^2b$.

9. Multiply $x^2 + 2x + 1$ by $2ax$.

10. Divide $5a^3b^3 + 15a^2b^4 + 10a^4b^2$ by $5a^2b^2$.

11. Divide $6ax + 9a^2x^2 + 12a^3x^3 + 15a^4x^4$ by $3ax$.

12. Four times and five times a number equals 108; what is the number? *Ans.* 12.

13. One-sixth of a pole is in the mud, four-fifths in the water, and 1 foot above the water; how long is the pole?

14. The sum of two numbers is 26, and their difference is 8; what are the numbers?

15. The sum of two numbers is 8, and their difference is 26; what are the numbers?

16. John has 9 marbles more than Harry, and both together have 35; how many has each?

17. Take $7a$ and $-4a$, and by adding first a positive and then a negative quantity to each, illustrate the following principles: —

(1) Adding on a positive quantity gives a sum larger (or smaller by a negative quantity) than the quantity to which it is added.

(2) Adding on a negative quantity gives a sum smaller (or larger by a negative quantity) than the quantity to which it is added.

18. Make examples to illustrate the following principles: —

(1) Subtracting a positive quantity gives a remainder smaller (or larger by a negative quantity) than the minuend.

(2) Subtracting a negative quantity gives a remainder larger (or smaller by a negative quantity) than the minuend.

19. From 0 subtract: $-a^2x$; a^3b; $3ac$; $x-y$; $a-2b$; $a^2-2ab+b^2$.

LESSON X.*

Multiplication by Negative Quantities.

Multiply:

1. -7 by 3. Prove by addition.

2. 9 by -4. Prove by addition.

3. -5 by -3. Prove by addition.

4. $-8ax^2$ by -1.

5. $25a^3x^2$ by $-y$.

6. $-5x^2y^2$ by $-a$.

7. $-7a^5x^2$ by $-2a^2y$.

8. $-4a^3b^2$ by $2abc$. **9.** a^2-2ab by $-a$.

10. $-3x^2-2x+4$ by $-3x$.

11. $6a^2c-3ac^2$ by -2.

12. $-7x-9$ by -1.

13. $8x^2-2y^2-3$ by -1.

14. a^2+3a^2b+7 by $-2a^3x$.

15. $7a^2-3ab+2c^2-3b^2c$ by $5abc$.

16. $12x^3y+7x^2y^2+14xy^3$ by $2xy$.

17. $b^3-a^3-c^3$ by $-abc$.

18. $m^4+n^4-r^4-s^4$ by -1.

19. $a^4-a^3-a^2+a-1$ by $-a$.

20. $7+3x-6x^2+3x^3$ by $2x^2$.

21. From 0 subtract each of the following quantities: —

a; $5a$; $-7x$; -3 four times; $-2x$ three times; $3ab$; $27xy$; $-4ax$; $a-b$; $4-x$.

22. To 0 add each of the following quantities: —

6; -7; $5a$; $12xy$; $-6xy$; x four times; $-x$ four times; ab minus three times; $-ab$ minus three times; $6a^2x$ minus twice; $3a-b$ twice; $2x-3y$ minus once; $a-b-c$ minus once.

23. Three-eighths of a number, three-sevenths of it, and 11 added together equals the whole number; what is the number?

24. I bought the same number of lead-pencils, penholders, and slate-pencils. For the lead-pencils I paid 3 cents each, for the penholders, 5 cents each, and for the slate-pencils a cent each. The whole cost \$1.08; how many did I buy of each?

25. A farmer sold a number of bushels of corn, twice as many bushels of wheat, and as many bushels of potatoes as of corn and wheat together. He got 40 cents a bushel for his corn, 80 cents a bushel for his wheat, and 50 cents a bushel for his potatoes. In all he received $105.00; how many bushels did he sell of each?

26. From $18x^2 - 3x + 27$ subtract $12x^2 + 4x - 3$.

27. From $3a^3b^2 - 5ab^2 + 6b^3$ subtract $3a^3b^2 - 4a^2b^2 - 3ab^2 + 9b^3 + 7$.

28. From $a^3 - b^3$ subtract $a^3 - 3a^2b + 3ab^2 - b^3$.

LESSON XI.*

Multiplication continued — Powers of Quantities.

Multiply:

1. a^4 by a^3.

2. a^4 by $-a^3$.

3. $-a^4$ by $-a^3$.

4. a by a.

5. $-a$ by $-a$.

6. $-x^2$ by $-x^3$.

7. $6ax$ by $-2a^3$

8. $-3m$ by $2n$.

9. $-x^2$ by x.

10. $a^2 + 2ab + b^2$ by $a + b$.

11. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

12. $a^3 - 9a^2 + 27a - 27$ by $a - 3$.

13. $a + b + c$ by $a + b - c$.

14. $a - b - c$ by $a - b + c$.

15. $2x^2 - 8x + 7$ by $x + 3$.

16. $3x^3 - 9x^2 + 2x - 4$ by $x + 2$.

17. $a^2 + b^2$ by $a + b$.

18. $3x^3 + 4x - 5$ by $x - 6$.

19. $x + 7$ by $x - 7$.

20. $x + c$ by $x - c$.

21. $3a^3 - 7ad + 2c - 5$ by $a + c$.

Expand:

22. $(a - b)^3$. **23.** $(x + 2)^2$. **24.** $(x - 5)^3$.

Multiply:

25. $x^2 - 2bx + b^2$ by $x^2 + 2bx + b^2$.

26. $x^3 - a^3$ by $x^3 + a^3$.

27. A drover bought the same number of sheep and cows; for the sheep he paid $3.00 each, and for the cows $27.00; for all he paid $270.00. How many did he buy of each? *Ans.* 9.

28. A drover bought a certain number of cows, the same number of oxen, and three times as many sheep; for the cows he paid $25.00 each, for the oxen $50.00 each, and for the sheep, $3.00 each; it cost him $7.00 to have them brought to market. The whole cost was $679.00; how many did he buy of each?

29. Multiply $3a$ by $-2ax$, and subtract the product from zero.

30. Multiply $a^2 - b^2$ by $-a$, and subtract the product from zero.

31. Multiply $a + 5$ by $a - 5$, and subtract the product from zero.

LESSON XII.*

Division.

Divide:

1. a^4 by a^2.

2. a^4 by $-a^2$.

3. $-a^4$ by $-a^2$.

4. $-a^4$ by a^2.

5. a^3x by $-a^2x$

6. $-a^3b^2y$ by a^2by.

7. $-8c^3d$ by $-4c^3$.

8. $24x^2$ by $6x$.

9. $-18m^4$ by $3m^4$.

10. $-27ad$ by $-27a$.

11. a^2-ab by a.

12. $6x^3-6ax^2$ by $6x^2$.

13. $5a^2y-10ay$ by $-5ay$.

14. $14a^5b^3-7a^4b^2$ by $-7a^4b^2$.

15. $12ax-36a^2x$ by $12ax$.

16. $45b^8-30b^6+15b^4$ by $-15b^4$.

17. $8a^3b^2c-16a^4b^3c^2$ by $-8a^3$.

18. $-7a^5-14a^3$ by $-7a^3$.

19. $a-b+c$ by -1.

20. $7-4a$ by -1.

21. Take $a^2+4ab-3ac$; multiply by a^4b^2; then divide by a^5b; then multiply by $-b^3c^3$; then divide by $-bc^2$; what is the last quotient?

22. Multiply $a^2-2ab+b^2$ by -1, and then divide the product by -1.

23. Multiply $a-b+c-d$ by $-a$, and divide the result by -1.

24. Multiply $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ by $a-b$, subtract $a^5-5a^4b+5ab^4-b^5$ from the product, and divide the remainder by $-10a^2b^2$.

25. Divide $-6a^2x^4+12a^3x^3-6a^4x^2$ by $-6a^2x^2$.

26. Divide $-ax^2y+bx^2y-cx^2y$ by $-x^2y$.

27. One-third, one-fifth, and one-ninth of a number equals 74 minus the number; what is the number?

28. 40 divided by a certain number gives 3 less than 64 divided by the same number; what is the number?

29. Multiply $a+b$ by $-a$ and subtract the product from 0.

30. From 0 subtract $-7a$; $6xy$; -36; $-8c$; -9; $3-x$; a^2-b^2.

31. Multiply $a-b$ by -1; by 1.

32. Divide $a-b$ by -1; by 1.

LESSON XIII.*

Division continued — Miscellaneous Examples.

Divide:

1. $-75a^3b^4$ by $-25a^2b^4$.

2. $125ax^2$ by $-25ax$.

3. x^2 by x. **4.** x^2 by $-x$.

5. $96b^2c-72b^3c-36ab^2c$ by $-12b^2c$.

6. $a^3+3a^2b+3ab^2+b^3$ by $a+b$; by $-a-b$.

7. $a^3-3a^2b+3ab^2-b^3$ by $a-b$; by $-a+b$.

8. $x^3-3x^2y+3xy^2-y^3$ by $x^2-2xy+y^2$.

9. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$ by $x^2+2xy+y^2$.

10. Divide the same quantity by $x+y$.

11. $3a^2+2a-8$ by $3a-4$.

12. x^2-5x+6 by $x-3$.

13. $6x^2-4x-42$ by $3x+7$.

14. $x^3+3x^2-25x+21$ by $x-3$.

15. $2a^2-3ax-2x^2$ by $2a+x$.

16. $6a^3+5a^2+9$ by $2a+3$.

17. $5x^3-23x^2+4$ by $5x+2$.

18. $4a^2b^2+4ab^2+b^2-c^2$ by $2ab+b+c$.

To find the value of x:

19. $-x = 8$. **20.** $2x - 5 = 4x - 31$.

21. A father is four times as old as his son, but in 14 years he will be only twice as old; what is the age of each? *Ans.* 7 and 28.

22. One man has worked three times as long as another, but if they continue to work 2 hours longer, he will have worked only twice as long; how long has each worked?

23. A man sold a certain number of horses at $100 each, and bought two more than the same number of cows for $30 each, and had $640 left; how many horses did he sell, and how many cows did he buy?

24. Multiply $3a + 4b$ by itself.

25. Multiply $2x - 3y$ by itself.

LESSON XIV.*

Division continued — Square Root and Cube Root of Monomials — The Signs ± and √.

Divide:

1. $35a^3b$ by $7ab$. **4.** $a^3 - b^3$ by $a - b$.

2. $-15x^3$ by $-5x^2$. **5.** $a^3 + b^3$ by $a + b$.

3. $a^2 - b^2$ by $a - b$. **6.** $a^4 - b^4$ by $a^2 - b^2$.

7. $a^4 - b^4$ by $a + b$; also by $a - b$.

8. $x^5 - y^5$ by $x - y$.

9. $x^5 + y^5$ by $x + y$. **10.** $x^6 + y^6$ by $x^2 + y^2$.

11. Multiply $a^3 + b^3$ by $a - b$, and divide the product by $a^2 - b^2$.

12. Multiply $a^3 + b^3$ by $a^3 - b^3$, and divide the product by $a^2 - b^2$.

13. The square roots of 1; 4; 9; 16; 25; 36; 49; 64; 81.

14. Square the following quantities: x; $-x^2$; x^3; a^2; $-a^5$; b^6; c^2d^2; $-a^2b^3$.

15. Cube the following quantities: a; a^2; $-a^3$; x^2; x^4; $-ax^2$; $-a^2x^3$; a^2b^2c; $-ab$.

16. Square roots of the following quantities: x^2; x^4; b^6; a^2b^4; $a^2b^2c^2$; x^8y^4.

17. Cube roots of the following quantities: a^9; a^3; $-b^{12}$; $-a^3x^3$; $-a^6x^3$; a^9b^{12}.

18. Square the following quantities: $2a$; $-3a^2$; $-3a^2x$; $4a^2x^3$; $-5x$; $8a^2y^2$.

19. Cube the following quantities: -5; $3x$; $2x^2$; $-4ax$; $5a^2x^3$; $-7ax^4$.

20. Square root of the following quantities: 64; $4x^2$; $25a^4$; $81a^2b^4$; $144c^2d^4$.

21. Cube roots of the following: $8a^3$; $27a^9$; $-125a^3b^6$; $-64x^9y^6$; $343a^3x^{12}$.

22. Henry had eight marbles more than John; five times the number John had equalled three times the number Henry had; how many had each?

23. Divide 20 into two such parts that six times the less equals four times the greater.

24. One-third of a number less 5 equals one-fourth of it; what is the number?

LESSON XV.

Involution — Binomial Theorem.

Expand:

1. $(a^2)^3$.
2. $(x^3)^4$.
3. $(b^4)^5$.
4. $(-a^2)^2$.
5. $(-a^2)^3$.
6. $(-b^2)^4$.
7. $(-ab)^2$.
8. $(ab)^2$.
9. $(-ab)^2$.
10. $(ab)^2$.
11. $(2x^2y)^5$.
12. $(5x^3y^2)^3$.
13. $(-2x^3y)^4$.
14. $(2x^3y)^4$.
15. $(-c^3)^4$.
16. $(-3ac^3)^3$.
17. $(-x)^2$.
18. $(4x^2y^3)^5$.
19. $(m+n)^2$.
20. $(m+n)^3$.
21. $(m-n)^2$.
22. $(m-n)^3$.
23. $(a-x)^5$.
24. $(x+y)^7$.
25. $(a+2)^3$.
26. $(b-3)^4$.
27. $(x-3)^3$.
28. $(a+b)^3$.
29. $(c+5)^2$.
30. $(c-6)^2$.

Note. For advanced work, see Lesson XXXII.

31. Five times the sum of a certain number and six, equals four times the sum of the same number and five; what is the number?

32. A farmer had a certain number of sheep, and his neighbor had ten more; the former sold his for $4.00 each, and the latter for $3.00, and each received the same sum for all; how many sheep had they?

33. Four-sevenths and three-fifths of a certain time added together equals the whole time plus 6 hours; how long is the time?

Divide:

34. x^3+6x^2-x-30 by $x+5$.

35. a^3+a^2-9a-9 by $a+1$.

36. $x^3 - 16x^2 - 3 + 48$ by $x^2 - 16$.

37. $x^4 - 9x^3 + 4x - 36$ by $x^3 + 9$.

Quick oral work: —

Expand: $(a+b)^2$; $(a+b)^3$; $(a-b)^3$; $(a-b)^4$; $(a+b)^4$; $(a+b)^5$; $(a-b)^5$.

$(x+y)^2$; $(x-y)^2$; $(x+y)^3$; $(x-y)^3$; $(x-y)^4$; $(x+y)^4$; $(x-y)^5$; $(x+y)^6$; $(x-y)^7$.

LESSON XVI.*

Relation of Quantities — The Parenthesis — Factoring.

ORAL EXERCISE.

Give the relation of:

1. cd to c and d.
2. $m + n$ to m and n.
3. $m - n$ to m and n.
4. $2xy$ to x and y.
5. 9 to 3; to -3.
6. $16a^2$ to $4a$; to $-4a$.
7. $3a$ to $9a^2$.
8. $25a^2b^4$ to $5ab^2$; to $-5ab^2$.
9. $14a$ to 7 and a.
10. $4a$ to a and $3a$.
11. $-5a$ to $-2a$ and $-3a$.
12. $3c$ to $-5c$ and $8c$.
13. $-3c$ to $-7c$ and $4c$.
14. $-5x$ to $-12x$ and $7x$.

15. $-12x^2$ to $4x$ and $-3x$.

16. $27x^2$ to $3x$ and $9x$.

17. $-x$ to $8x$ and $-9x$.

18. x to $7x$ and $-6x$.

19. $3x$ to $7x$ and $4x$.

20. a^2x^2 to ax; to $-ax$.

21. $7a^2x$ to $9a^2x$ and $-2a^2x$.

22. $-18a^4x^2$ to $9a^2x$ and $-2a^2x$.

23. $-45a^3b$ to $15a^3$ and $-3b$.

24. $18ax^2$ to $9ax$ and $2x$.

25. $8b$ to 4 and b.

26. $10x$ to 5 and x.

Factor:

27. $a^3x + a^3y - a^3$.

28. $6x^2y - 12xy^2$.

29. $5x^3y^2 + 10x^2y^3 - 5x^2y^2$.

30. $a^3 + a^2b + a^2c$.

31. $7ax^2 + 14ax$.

32. $18 - 27ab + 9a$.

33. $7a - 14ax + 21ax^2$.

34. $15a^2b^3 - 10a^3b^2 + 25a^2b^2$.

35. $6a - 12a^2 + 18a^3$.

Combine:

36. $(x - 1)(x + 1)$.

37. $(x + 2a)(x - 2a)$.

38. $(2ax + 3by)(2ax - 3by)$.

39. $(3a^2b - 5ac)(3a^2b + 5ac)$.

40. $(a^2 + b^2)(a^2 - b^2)$.

Factor:

41. $c^2 - d^2$.

42. $m^4 - n^2$.

43. $4a^2 - 9b^2$.

44. $25m^2 - 16n^2$.

45. $81a^2b^2 - 36$.

46. $625a^6 - b^2$.

47. $64b^4 - 4$.

48. $4a^2 - 1$.

49. $1 - 9b^2$.

50. $a^4b^2 - 1$.

Separate into three factors:

51. $ax^2 - ay^2$.

52. $5a^2b^2 - 5x^2y^2$.

53. $x^4 - y^4$.

54. $16x^4 - 81y^4$.

Add:

55. $a(a - 5)$ to $5(a - 5)$.

56. $2x(x + 6)$ to $7(x + 6)$.

Subtract:

57. $6(m + 6)$ from $m(m + 6)$.

58. $b(a + b)$ from $a(a + b)$.

59. If a number be subtracted from 25, and also 14 be subtracted from the number, five times the first remainder equals six times the second; what is the number?

60. John and Henry had the same number of marbles; John gained 7 and Henry lost 8, then John had twice as many as Henry; how many had each in the beginning?

ORAL EXERCISE.

Combine: $(x + 3)(x - 3)$; $(a - 7)(a + 7)$; $(3a - 5b)(3a + 5b)$; $(2xy - 3a)(2xy + 3a)$; $(2x^2 - 2y)(2x^2 + 2y)$; $(5x^3 - ab)(5x^3 + ab)$.

Expand: $(x + 3)^2$; $(x + 2)^2$; $(x - 4)^2$; $(x - 7)^2$.

LESSON XVII.

The Parenthesis — Problems to be Solved Inspection.

Remove parentheses:

1. $a + (a - b)$.

2. $a - (a - b)$.

3. $9 + (x - 4)$.

4. $9 - (x - 4)$.

5. $x + (a - b + c)$.

6. $a^3 - (a^2 - a + 1)$.

7. $12 - (7 - a + c)$.

8. $15 - (a^2 - a^3 + 15)$.

Remove parentheses and simplify:

9. $6a^2 + 3a - 2(b - c + d)$.

10. $9a^2 + 5a - 2(a^2 + 2a)$.

11. $5x^2 + 9ax - 4x - 4x(x + 2a - 1)$.

12. $8a^3 + 7a^2 - 8(a^3 - a^2 + 1)$.

13. $2x^2 + (3x^2 - x) - 4(x^2 - 2x - 1)$.

14. $a^3 - 2(a^2 - 3a + 2) + 2(2a + 2)$.

15. $a^3 - (a^2 + a) + 2(a^3 + a^2) - 3(a + 1)$.

Combine by inspection:

16. $(3a - 2x)(3a + 2x)$.

17. $(a^2 - b^2)(a^2 + b^2)$.

18. $(5a - 3b)(5a + 3b)$.

19. $(2ax + 3by)(2ax - 3by)$.

20. $(6x^2 + 5y^2)(6x^2 - 5y^2)$.

21. $(2a^2b^2 - 5)(2a^2b^2 + 5)$.

22. $(x + 1)(x - 1)$.

23. $(xy - 1)(xy + 1)$.

24. $(1 + a)(1 - a)$.

Factor:

25. $a^2 - y^2$.

26. $4a^2 - 9$.

27. $25 - 9a^2b^2$.

28. $3x^2 - 12y^2$.

29. $64a^2x^2 - 36b^2x^2$.

30. $16a^4 - 81b^4$ (three factors).

31. $25m^2 - 16n^2$.

32. $625a^4 - 9b^4$.

33. $a^4 - b^2$.

34. $49a^6 - 4b^2$.

35. $81a^4 - 36a^2$.

36. $16a^2 - 1$.

37. $1 - 4x^2$.

38. $1 - 9x^4$.

39. $25x^6 - 1$.

40. $x^4 - 1$.

41. $1 - x^4$.

42. $625a^2b^2 - 4x^2y^2$.

43. $9a^4x^4 - 4b^2y^2$.

44. $a^4b^2 - 4$.

Expand:

45. $(x + y)^4$.

46. $(x - y)^4$.

47. $(x + 3)^3$.

48. $(x - 4)^3$.

49. $(x - 5)^3$.

50. $(x + 3)^2$.

Factor together the two first terms and the two last terms, and simplify:

51. $ax + bx - ay - by$. *Ans.* $(x - y)(a + b)$.

52. $3ab + 3ac + 4bx + 4cx$.

Ans. $(3a + 4x)(b + c)$.

53. $10 - 5a - 6b + 3ab$.

54. $3ax - xb + 12a - 4b$.

55. $2ab + 3bc - 10a - 15c$.

56. $6ax - 10ay - 9bx + 15by$.

Ans. $(3x - 5y)(2a - 3b)$.

LESSON XVIII.

Multiplication at Sight — Factoring — Problems in which Two Letters are Used to Indicate the Unknown Quantities.

Expand at sight:

1. $(x + 2)^2$.
2. $(x + 7)^2$.
3. $(x + 4a)^2$.
4. $(x + 3c)^2$.
5. $(a + 3b)^2$.
6. $(a + 5b)^2$.
7. $(x + 1)^2$.
8. $(x + 9)^2$.

Factor by inspection:

9. $x^2 + 14x + 49$. *Ans.* $(x + 7)^2$.
10. $x^2 + 20x + 100$. *Ans.* $(x + 10)^2$.
11. $x^2 + 4ax + 4a^2$. *Ans.* $(x + 2a)^2$.
12. $x^2 + 6ax + 9a^2$.
13. $x^2 + 8x + 16$.
14. $a^2 + 2a + 1$.
15. $b^2 + 10bx + 25x^2$.

Expand at sight:

16. $(x - 7)^2$.
17. $(y - 3)^2$.
18. $(a - 3b)^2$.
19. $(b - 15)^2$.
20. $(x - 3a)^2$.

Factor by inspection:

21. $x^2 - 18x + 81$. *Ans.* $(x - 9)^2$.
22. $x^2 - 12x + 36$.
23. $x^2 - 24x + 144$.
24. $a^2 - 30a + 225$.
25. $a^2 - 22a + 121$.

Multiply at sight:

26. $x + 1$ by $x + 4$. *Ans.* $x^2 + 5x + 4$.
27. $x - 3$ by $x - 5$. *Ans.* $x^2 - 8x + 15$.

28. $x + 1$ by $x + 2$.

29. $x + 2$ by $x + 5$.

30. $x + 8$ by $x + 3$.

31. $x - 4$ by $x - 6$.

32. $x - 2$ by $x - 4$.

33. $x - 7$ by $x - 1$.

34. $x + 2a$ by $x + 5a$.

35. $x + 3a$ by $x + 7a$.

36. $x - 4a$ by $x - a$.

Factor by inspection:

37. $x^2 + 4x + 3$.

38. $a^2 - 5a + 6$.

39. $x^2 + 3x + 2$.

40. $x^2 - 10x + 16$.

41. $x^2 + 12x + 11$.

42. $x^2 + 5x + 4$.

43. $x^2 - 4x + 3$.

Multiply:

44. $5 + x$ by $7 + x$.

45. $2 + y$ by $5 + y$.

Factor:

46. $15 + 8x + x^2$.

47. $12 + 7x + x^2$.

ORAL EXERCISE.

48. Find two numbers:

1. Whose sum is 5, and product 6.
2. Whose sum is 4, and product 3.
3. Whose sum is -8, and product 15.
4. Whose sum is -3, and product -10.
5. Whose sum is 4, and product -21.
6. Whose sum is 12, and product 35.
7. Whose sum is -12, and product 45.
8. Whose sum is $7a$, and product $-30a^2$.
9. Whose sum is $-8b$, and product $-20b^2$.
10. Whose sum is $-15c$, and product $56c^2$.
11. Whose sum is 7, and product 12.
12. Whose sum is -7, and product -44.

49. The sum of two numbers is 13 and the difference 3; what are the numbers?

50. Four times one number plus twice a second is 34, and twice the sum of the two is 22; what are the numbers?

51. Five times a number plus three times a second is 41, and three times the first minus three times the second is 15; what are the numbers?

LESSON XIX.

Factoring Arranged by Cases.

CASE I. General problem: $ab \pm ac = a(b \pm c)$.

Factor:

1. $a^3b - 3a^2c$.

2. $7abx + 14aby$.

3. $5x^2 + 10x^3$.

4. $25a^2b^3 - 5a^2b$.

5. $13a^2 + 13a$.

6. $9a^2b + 18a^2c - 27a^2d$.

7. What are the peculiar features of Case I.?

CASE II. General problem: $a^2 - b^2 = (a+b)(a-b)$.

Factor:

1. $4 - 9a^2$.

2. $25a^2 - 16b^2$.

3. $64a^2x^2 - 100b^2y^2$.

4. $1 - 4x^2$.

5. $81y^4 - 1$. (3 factors.)

6. $121 - 4a^2b^2c^2$.

7. What are the peculiar features of Case II.?

CASE III. General problem: $a^2 + 2ab + b^2 = (a+b)(a+b)$.

Factor:

1. $m^2 + 2mn + n^2$.
2. $4m^2 + 12mn + 9n^2$.
3. $64b^2 + 32bx + 4x^2$.
4. $25a^2b^2 + 20abx + 4x^2$.
5. $16x^2 + 8x + 1$.
6. $1 + 4x^2 + 4x^4$.
7. What are the peculiar features of Case III.?

Case IV. General problem: $a^2 - 2ab + b^2 = (a - b)(a - b)$.

Factor:

1. $36a^2 - 36ab + 9b^2$.
2. $x^2 - 10x + 25$.
3. $49 - 28c + 4c^2$.
4. $100x^2y^2 - 100xy + 25$.
5. $5a^2 - 10ab + 5b^2$. (3 factors.)
6. $81x^4 - 72xy + 16y^2$. (4 factors.)
7. What are the peculiar features of Case IV.?

Case V. General problem: $(a + b)(a + c) = a^2 + (b + c)a + bc$.

Factor:

1. $x^2 + 5x + 6$.
2. $y^2 + 3y + 2$.
3. $m^2 + 13m + 40$.
4. $b^2 + 15b + 54$.
5. $7x^2 + 56x + 105$. (3 factors.)
6. $3c^2 + 18c + 24$.
7. What are the peculiar features of Case V.?

Case VI. General problem: $(a - b)(a - c) = a^2 - (b + c)a + bc$.

Factor:

1. $x^2 - 7x + 6$.
2. $a^2 - 11x + 10$.
3. $x^2 - 17x + 30$.
4. $x^4 - 12x^2 + 32$.

5. $x^4 - 9x^2 + 20$.
6. $x^2 - 8ax + 15a^2$.

7. What are the peculiar features of Case VI.?

Combine by inspection:

1. $(a + 9)(a - 8)$.
3. $(x + 2a)(x - 4a)$.
2. $(x + 4)(x - 8)$.
4. $(x + 8b)(x - b)$.

CASE VII. General problem: $(a + b)(a - c) = a^2 + (b - c)a - bc$.

Factor:

1. $x^2 + 3x - 10$.
6. $a^2 - 2a - 63$.
2. $y^2 - 5y - 14$.
7. $a^2 + 2a - 63$.
3. $x^2 - 11x - 12$.
8. $x^2 - 3x - 40$.
4. $a^2 + a - 56$.
9. $x^2 - 3ax - 10a^2$.
5. $a^2 - a - 56$.
10. $x^2 + 3cx - 4c^2$.

What are the peculiar features of Case VII.?

For Cases VIII. and IX., see Lesson XXI.

1. A and B have herds of cattle of which A's is the larger; the number in both herds is 95, and three times the number A has minus the number B has is 117; how many had each? *Ans.* A, 53; B, 42.

2. $\frac{x}{3} + \frac{y}{2} = 17$. $\frac{x}{4} + \frac{y}{3} = 12$.

LESSON XX.*

Common Divisor — Greatest Common Divisor, G. C. D.

IN the following examples find two common divisors, one of which will be the greatest (integral) divisor possible. Also quotients, when the quantities are divided by it.

1. 36, 48, 72, 96. **3.** $9x^2y$, $27xy$, $45x^3y^3$.
2. $5a^2b$, $10a^3b^2$, $15a^2b^2x$. **4.** a^2c, a^2d, a^3, $5a^2$.
5. a^2x^2y, $a^2x^2 - 5a^3x$, $abx + acx$.
6. $x^2 - y^2$, $ax - ay$, $x^2 - 2xy + y^2$.
7. 6, 7, 5, 11.

Find G. C. D. only:

8. $7a + 7b$, $ax + ab$, $a^2 - b^2$.
9. $12b - 24$, $bx - 2x$, $x^2 - 4$.
10. $a^2x - 3a^3$, $3ax^2 - 27a$, $ax - 3a$.
11. $x^2 - y^2$, $x^2 - 2xy + y^2$, $ax^2 - axy$.
12. $3a^2 - 3c^2$, $a^2 + 2ac + c^2$, $5ab + 5bc$.
13. $m^2 - 16$, $m^2 + 8m + 16$, $mx + 4x$.
14. $36 - a^2$, $36 + 12a + a^2$.
15. $81 - a^4$, $9 + 6a + a^2$.
16. $x^2 - 25$, $x^2 + 8x + 15$.
17. $x^2 - 49$, $x^2 - 9x + 14$, $3x + 21$.
18. $y^2 + 6y + 9$, $y^2 - 9$, $y^2 + 2y - 3$.
19. $a^2 - 3a - 10$, $a^2 - 10a + 25$.
20. $9a^2 - 25b^2$, $15ax - 25bx$.
21. $c^2 - 8c + 15$, $c^2 - c - 20$, $2c - 10$.

Find the value of x or y:

22. $\frac{1}{4} - \frac{1}{5} = \frac{1}{x}$. *Ans.* $x = 20$.
23. $\frac{1}{3} + \frac{1}{5} = \frac{1}{x}$. *Ans.* $x = 1\frac{7}{8}$.
24. $\frac{1}{3} + \frac{2}{5} = \frac{1}{y}$. **25.** $\frac{x}{4} + \frac{x}{5} = 9$.

26. One man can do a piece of work in 3 hours and another in 6 hours; how long will it take them both to do it?

27. Two pipes discharge water into a cistern; one will fill it in 8 hours and the other in 4 hours; how long will it take both to fill it?

28. Of two pipes one will fill a tank in 4 hours, and both in 3 hours; how long will it take the other to fill it?

29. In a school of boys and girls, two times the number of boys plus five times the number of girls equals 100, and four times the number of boys minus five times the number of girls equals -10; how many are there of each?

30. Divide $a^7 + b^7$ by $a + b$.

31. Multiply $a^4 - a^3b + a^2b^2 - ab^3 + b^4$ by $a + b$.

32. $m + (n - r + 3)$; remove parentheses.

LESSON XXI.

Cases of Factoring, Continued — G. C. D.

Divide:

1. $x^3 + y^3$ by $x + y$.

2. $x^3 - y^3$ by $x - y$.

3. $y^3 + 8$ by $y + 2$.

4. $y^3 - 8$ by $y - 2$.

5. $x^5 + y^5$ by $x + y$.

6. $x^5 - y^5$ by $x - y$.

FACTORING. CASE VIII. General Problems.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Factor:

1. $m^3 + n^3$.
2. $c^3 + d^3$.
3. $(a^2)^3 + (b^2)^3$.
4. $a^3 + 27$.
5. $x^3 + 125$.
6. $x^6 + 64$.
7. What are the peculiar features of Case VIII.?

Case IX. General Problems.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$
$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

Factor:

1. $r^3 - s^3$.
2. $e^3 - m^3$.
3. $(m^2)^5 - (n^2)^5$.
4. $x^3 - 343$.
5. $125 - a^3$.
6. $27 - b^6$.
7. What are the peculiar features of Case IX.?

For Case X., see Lesson XXIII.

Find the greatest common divisor (G.C.D.) of the quantities in each of the following:

1. a^2b, ab^2, and $-a^2b^2$.
2. $3xy$, $9ax$, $-3bx^2$, and $12cx$.
3. $7ax + 7bx$ and $3a^2 - 3b^2$.
4. $ax - ay$, $x^2 - y^2$, and $x^2 - 2xy + y^2$.
5. $a^2 + 2ab + b^2$ and $a^2 - b^2$.
6. $a^2 - b^2$ and $a^3 + b^3$.
7. $x^2 - 4$ and $x^2 - 4x + 4$.
8. $9 - a^2$ and $9 - 6a + a^2$.
9. $x^2 - a^2$, $x^2 + (a + b)x + ab$, and $5x + 5a$.
10. $x^2 - 4x + 4$ and $x^2 + x - 6$.

Problems in division.

1. Divide $(x + y)^3$ by $(x + y)^2$.
2. Divide $(a + b)^5$ by $(a + b)^3$ and expand the quotient.

3. Divide $(x + 3)^5$ by $(x + 3)^2$ and expand the quotient.

4. The G. C. D. of $(c + d)^4$, $(c + d)^3$, and $(c + d)^5$.

Miscellaneous examples for factoring.

1. $x^2 + 5x + 4$.
2. $x^2 - 12x + 32$.
3. $x^2 - 3x - 108$.
4. $x^2 - x - 132$.
5. $x^2 + x - 72$.
6. $x^2 - 2x - 63$.
7. $25x^2 - 9c^2$.

LESSON XXII.*

Fractions — Use of the G. C. D. — Axioms.

Reduce to lowest terms:

1. $\frac{3ax^2}{6ax}$.
2. $\frac{16a^3b^2}{8a^4b^3}$.
3. $-\frac{5ax^3y^2}{25ax^4y^2}$.
4. $\frac{18m^2y^2}{9m^2y^2}$.
5. $-\frac{17a^2b^2}{34a^3b^3}$.
6. $\frac{39x^4y^4}{13x^2y^4}$.
7. $\frac{a^3b + a^4}{3a^3}$.
8. $\frac{a^3b + a^4}{a + b}$.
9. $\frac{3x^3 - 3x^2y}{6ax^2}$.
10. $\frac{ax^3 + ax^2y}{ax + ay}$.
11. $\frac{x^2 - y^2}{x^2 - 2xy + y^2}$.
12. $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$.
13. $\frac{a^3 + b^3}{a^2 - b^2}$.
14. $\frac{4a(x^3 - y^3)}{8a^2(x^2 - y^2)}$.

15. $\frac{x^2 + 7x + 12}{x^2 + 9x + 20}$.

16. $\frac{a^2 - 6a + 5}{a^2 - 3a - 10}$.

17. $\frac{x^2 - 16}{x^2 + 8x + 16}$.

18. $\frac{x^3 - 27}{x^2 - 6x + 9}$.

19. Give the axiom by which you find the value of x in each of the following:—

1. $x + 6 = 9$.
2. $x - 3 = 7$.
3. $3x = 15$.
4. $\frac{x}{5} = 2$.

20. $2x + 3y = 35$.
$3x + y = 35$.

21. $3x + y = 33$.
$4x - 3y = 57$.

What axioms have you used in solving the above?

Use two letters for the unknown quantities in the following:—

22. A cistern holding 64 bbls. of water, is filled by two pipes; both running 8 hours will fill it, or No. 1 running 7 hours and No. 2 running 5 hours will fill it; how many bbls. per hour does each discharge into the cistern?

Interpret the negative result.

23. A and B are together worth \$3000; but twice A's property added to B's would be \$6500; what is each worth?

Find the value of x in terms of a, b, and c:

24. $x + b = a$.

25. $x - ab = a^2$.

26. $ax + ab = a^2$.

27. $ax + b = c$.

28. $\frac{x}{b} + b = \frac{a^2}{b}$.

29. $ax + bx = c$.

30. Expand $(x + b)^9$.

31. Expand $(x - c)^{10}$.

32. Expand $(2x - 5b)^8$.

Factor:

33. $81x^2 - 25y^2$.

34. $x^2 + 16ax + 64a^2$.

35. $x^2 - 18abx + 81a^2b^2$.

36. $x^2 - 18x - 17$.

37. $a^2 + 17a - 30$.

LESSON XXIII.*

Least Common Multiple, L. C. M. — Factoring, Case X.

IN each of the following examples find two common multiples, one of which will be the least common multiple, L. C. M.

1. 7, 21, 35, and 15. Prove.

2. a^2, ab^2, a^2c^2, b^2c^2. Prove.

3. x^2, y^2, $3x$, $5y$.

4. $2x$, $3y$, $2y(a+b)$.

5. $5m$, $3mn$, $5n$.

Find the least common multiple:

6. $3a$, $4ax$, $2a^2x^2$, $12a^2x$. Prove.

7. $16am$, $-8m^2$, $12a^2$, $-3a^2m^2$.

8. $a+b$, $a-b$.

9. $x+3$, $x-3$, x^2-9.

10. a^2, $b+2$, b^2-4.

11. $3ab$, $3(a+b)$, $2(a-b)$.

12. a^2-b^2, $a^2-2ab+b^2$, $a(a-b)$.

13. a^2-b^2, $a^2+2ab+b^2$, a^2+ab.

14. $x+3$, $x-3$, x^2-9.

15. $x+2$, $x-5$, $x^2-3x-10$.

16. x^2+4x+3, x^2+x-6, $ax+3a$.

17. $x^2 - 25,\ x^2 + 2x - 35,\ ax - 5a.$

18. $x^2 - 2x - 8,\ x^2 - 4,\ x^2 - 6x + 8.$

19. $a^2 - b^2,\ a^3 - b^3,\ a^2 - ab.$

20. $a^2 - b^2,\ a^3 + b^3,\ a^2 + 2ab + b^2.$

Add the following:

21. $5(x + 6),\ 4(x + 6),\ -3(x + 6),\ -7(x + 6).$

22. $3(a + b),\ -4(a + b),\ 5(a + b),\ 9(a + b),\ -2(a + b).$

23. $a(c + d),\ b(c + d).$ **24.** $a(x - y),\ -b(x - y).$

25. $a(a - b),\ -b(a - b).$

26. $a(a + b - 3),\ b(a + b - 3).$

Factoring. Case X. General Problems.

$$ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d).$$

$$ac - ad + bc - bd = a(c - d) + b(c - d) = (a + b)(c - d).$$

$$ac - ad - bc + bd = a(c - d) - b(c - d) = (a - b)(c - d).$$

Factor:

27. $ax + ay + dx + dy.$ **28.** $ax - 3a + 5x - 15.$

29. $a^3 - 3a^2 - 7a + 21.$

30. $6ax + 9ay - 4bx - 6by.$

31. $am - 2bm - an + 2bn.$

32. $a^2c^2 - abc^2 - a^2bc + ab^2c.$

Note. For Case XI., see Lesson XXXV.

Find value of x.

33. $x - (60 - x) = 20.$ **34.** $2x - (x - 7) = 22.$

35. $x - (8 - 2x) = 1.$

36. $\frac{1}{2}+\frac{1}{8}=\frac{5}{x}$.

37. $\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$.

38. A can mow a field in 4 hours, and B in 3 hours; how long will it take both? *Ans.* $1\frac{5}{7}$ hours.

39. A and B together can mow a field in 5 hours, and A alone can do it in 8 hours; how long will it take B to do it?

Reduce to lowest terms:

40. $\frac{x^2-y^2}{x^3+y^3}$.

41. $\frac{x^2-xy}{x^3-y^3}$.

42. $\frac{x^2-x-42}{x^2+x-56}$.

43. $\frac{a^2-2a+1}{a^2+5a-6}$.

LESSON XXIV.*

Use of the L. C. M. — Addition and Subtraction of Fractions.

Change the following fractions to a common denominator:

1. $\frac{a}{3}$ and $\frac{a}{4}$.

2. $\frac{a}{b}$ and $\frac{a}{c}$.

3. $\frac{a+b}{a}$ and $\frac{a+b}{a-b}$.

4. $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

Add the following:

5. $\frac{x}{2}$, $\frac{x}{3}$.

6. $\frac{x}{a}$, $\frac{x}{b}$.

7. $\frac{a+2}{3}$, $\frac{a+5}{4}$.

8. $\frac{x+5}{5}$, $\frac{x-3}{3}$.

9. $\frac{a+b}{a}, \frac{a+b}{b}.$

11. $\frac{7}{a+b}, \frac{7}{a}.$

10. $\frac{x+6}{3}, \frac{x-7}{4}.$

12. $\frac{3}{x+2}, \frac{4}{x}.$

13. $\frac{a+b}{a-b}, \frac{a-b}{a+b}.$

14. $\frac{x+2}{x-2}, \frac{x-2}{x+2}.$

15. $\frac{x+3}{x+1}, \frac{x-6}{x-1}.$

Subtract:

16. Quantities as in 13, second from first.

17. Quantities as in 14, second from first.

18. Quantities as in 15, second from first.

19. From $\frac{x}{a}$ take $\frac{x}{b}$.

20. From $\frac{a}{b}$ take $\frac{b}{a}$.

21. From $\frac{x}{3}$ take $\frac{x}{5}$.

22. From $\frac{3x+7}{3}$ take $\frac{4x+9}{4}$.

23. From $\frac{2a+8}{5}$ take $\frac{3a-7}{6}$.

24. From $\frac{a+4}{5}$ take $\frac{a+3}{6}$.

25. From $\frac{a+b}{3}$ take $\frac{a+b}{6}$.

26. From $\frac{a+b}{a}$ take $\frac{a+b}{b}$.

27. From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$.

28. From $\frac{a^2+b^2}{a^2-b^2}$ take $\frac{a-b}{a+b}$.

Unite the following as indicated by signs:

29. $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$

30. $\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{a^2+b^2}{a^2-b^2}.$

31. $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{4-x^2}.$

32. $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}.$

33.

$$\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}.$$

34.

$$\frac{1}{(x+y)(x+z)}-\frac{1}{(y+z)(x+y)}+\frac{1}{(x+z)(y+z)}.$$

35.

$$\frac{ac}{(a-b)(b-c)}+\frac{bc}{(a-b)(c-a)}+\frac{ab}{(b-c)(c-a)}.$$

36.

$$\frac{c+a}{(a-b)(a-c)}+\frac{a+b}{(b-c)(b-a)}+\frac{b+c}{(c-a)(c-b)}.$$

Find the value of x:

37. $3x-(100+x)=0.$

38. $ax-(bx+c)=a-b-c.$ *Ans.* $x=1.$

39. The sum of two quantities is a and the difference b; what are the quantities?

40. A grocer has tea of two prices; 5 lbs. of the higher priced and 4 of the lower are worth $6.27; 2 lbs. of each are worth $1.76; find the price of each kind.

Reduce to lowest terms. See Case X., factoring.

41. $\frac{2ab+5b+6a+15}{b^2+4b+3}$. *Ans.* $\frac{2a+5}{b+1}$.

42. $\frac{ax+3a-bx-3b}{a^2-2ab+b^2}$.

43. $\frac{3ax-ay-6x+2y}{a^2+2a-8}$.

LESSON XXV.*

Mixed Numbers to Fractions — Fractions to Mixed Numbers.

Change to a fraction:

1. $1+\frac{b}{a}$.

2. $3a+\frac{3b^2}{a}$.

3. $2x-\frac{a}{b}$.

4. $2x-a+\frac{a}{3}$.

5. $4a+\frac{ab}{a+b}$.

6. $a+b-\frac{a^2+b^2}{a+b}$.

7. $a-b+\frac{4ab}{a-b}$.

8. $x+3+\frac{9}{x-3}$.

9. $x-5+\frac{25}{x+5}$.

10. $a+b-\frac{a^2+b^2}{a-b}$.

11. $x-8+\frac{15x}{x-7}$.

12. $x+3-\frac{x+12}{x-4}$.

Change to a mixed number:

13. $\frac{x^2+9x+17}{x}$.

14. $\frac{2a^2-4a+8}{2a}$.

15. $\frac{a^2-ab+b^2}{a}$.

16. $\frac{3x^3-7x^2+4x+3}{x-2}$.

17. $\frac{5x^2+12x-41}{x^2+6}$. **18.** $\frac{a^2+b^2}{a+b}$.

19. $\frac{a^2+ab+b^2}{a+b}$.

20. $\frac{15x^3+36x^2+8x-3}{5x+2}$.

21. $\frac{6a^3-2a^2+2a-7}{3a^2+2a+3}$.

Add:

22. $\frac{a}{a+b}$ and $\frac{b}{a+b}$. **23.** $\frac{x}{x+3}$ and $\frac{x}{x-3}$.

24. $\frac{7}{a+b}$ and $\frac{4}{a^2-2ab+b^2}$.

25. Subtract $\frac{a}{a+2}$ from $\frac{a}{a+1}$.

26. Subtract $\frac{a-b}{a}$ from $\frac{a-b}{b}$.

Reduce to lowest terms:

27. $\frac{c^2-d^2}{c^2-2cd+d^2}$. **28.** $\frac{x^2+7x+12}{x^2-3x-28}$.

29. $\frac{5a^3+10a^2b}{5(a^4-b^4)}$.

30. A farmer had b sheep; he lost a third of them, and then he lost a third of the remainder; how many did he have left?

31. Henry had $5a$ marbles; he lost half of them, and then he lost $\frac{3a}{2}$; how many had he left?

32. Simplify:

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(a-c)(b-c)}+\frac{1}{(b-c)(a-b)}.$$

LESSON XXVI.*

Multiplication and Division of Fractions by Whole Quantities.

Multiply:

1. $\frac{a}{b}$ by b.

2. $\frac{a^2b}{3c^2}$ by $3ac$.

3. $\frac{2x^2y}{3am}$ by $3my$.

4. $\frac{x^2y^2}{5ab}$ by $10xy$.

5. $\frac{a}{b}$ by $3ax$.

6. $\frac{7a^3b^2}{9c^2d}$ by $18bc$.

7. $\frac{3a}{a+b}$ by $2b$.

8. $\frac{xy}{x-y}$ by xy.

9. $\frac{ab}{a-b}$ by $2(a-b)$.

10. $\frac{3ax}{2(x+y)}$ by x^2-y^2.

11. $\frac{a-b}{a+b}$ by a^2-b^2.

12. $\frac{x+5}{x^2-5x+6}$ by $x^2-7x+10$.

13. $\frac{x+a}{x^2+3ax-28a^2}$ by $x^2-5ax+4a^2$.

Divide:

14. $\frac{a}{b}$ by a.

15. $\frac{x^3y}{2a}$ by $3x^2a$.

16. $\frac{3ab^2}{2c}$ by $3abc$.

17. $\frac{5a^2b}{4c^2}$ by $5a^3b^2$.

18. $\frac{x}{5}$ by $2x$.

19. $\frac{12a^2c^2}{bm}$ by $4ac^2m$.

20. $\frac{a+b}{a}$ by $a(a+b)$.

21. $\frac{(x-y)^2}{xy}$ by $2(x-y)$.

22. $\frac{ab}{a-b}$ by $a(a+b)$.

23. $\frac{a(x-y)}{b}$ by $b(x-y)$.

24. $\dfrac{a^2 - 2ab + b^2}{c}$ by $a^2 - b^2$.

25. $\dfrac{a^2 + 2ab + b^2}{5}$ by $2(a^2 - b^2)$.

26. $\dfrac{a^3 + b^3}{a - b}$ by $a^2 - b^2$.

27. $\dfrac{x^2 - 2x - 35}{x + 2}$ by $x^2 + 7x + 10$.

28. $\dfrac{a^2 + x - 20}{x - 1}$ by $x^2 + 6x + 5$.

Express:

29. $2\frac{1}{2}$ times $x - 6$.

30. $3\frac{3}{4}$ times $x + 5$.

31. $7\frac{5}{8}$ times $x + 3$.

32. $4\frac{3}{5}$ times $x - 8$.

Multiply:

33. a^2 by $\dfrac{b}{a}$.

34. $\dfrac{2x}{y}$ by -1.

35. $3xy$ by $\dfrac{2}{xy}$.

36. $-\dfrac{2a}{b}$ by -1.

37. If a boat's crew can row x miles an hour, how far down a river flowing at the rate of 4 miles an hour can they go in one hour? In 2 hours? In $2\frac{1}{3}$ hours?

38. How far could the same crew row up the same river in 1 hour? In 3 hours? In $4\frac{1}{2}$ hours? In 5 hours and 40 minutes?

39. It took this same boat's crew 3 hours to row up the river a distance that they rowed down in one hour; what would be their rate per hour in still water?

40. A boat's crew rowed for 2 hours up a stream flowing at the rate of 2 miles an hour; returning it took them only 1 hour and 12 minutes; how fast can they row?

41. Simplify $\dfrac{x + 2}{x - 2} - \dfrac{x - 2}{x + 2} + \dfrac{x - 1}{x^2 - 4}$.

LESSON XXVII.*

Multiplying and Dividing Fractions and Mixed Quantities by Fractions.

Multiply:

1. $\frac{a}{b}$ by $\frac{a}{b}$.

2. $\frac{a}{b}$ by $\frac{b}{a}$.

3. $\frac{3a^2x}{5by}$ by $\frac{10by^2}{21a^2x^2}$.

4. $-\frac{7x^2y^2}{8a^3b}$ by $-\frac{8a^3}{7x^2y}$.

5. $\frac{x}{y}$ by $\frac{y^2}{x^2}$.

6. $-\frac{4x}{5a}$ by $-\frac{1}{4}$.

7. $\frac{a^2-b^2}{x^2-4}$ by $\frac{x-2}{a+b}$.

8. $\frac{a^3+b^3}{ab}$ by $\frac{b}{a^2-ab+b^2}$.

9. $\frac{9-a^2}{3+x}$ by $\frac{3+x}{3-a}$.

10. $\frac{a(a-b)}{a^2+2ab+b^2}$ by $\frac{a^2-b^2}{ab}$.

11. $1+\frac{x}{a}$ by $1-\frac{x}{a}$.

12. $3a-\frac{3b^2}{a}$ by $\frac{a}{3(a+b)}$.

13. $\frac{x^2-9x+20}{x^2+7x+6}$ by $\frac{x^2+2x+1}{x^2-8x+16}$.

14. $\frac{a^2+9a-10}{a^2-10a-11}$ by $\frac{a^2+2a+1}{a^2-2a+1}$.

15. $a-5-\frac{14}{a}$ by $\frac{a}{a^2-9a+14}$.

16. $\frac{x+2}{x+3}$ by $\frac{x+4}{x+5}$.

17. $\dfrac{x+4}{x-3}$ by $\dfrac{x-4}{x+3}$.

Divide:

18. $\dfrac{a}{b}$ by $\dfrac{a}{b}$.

19. $\dfrac{x}{y}$ by $\dfrac{y}{x}$.

20. $\dfrac{a}{2b}$ by $-\dfrac{1}{2}$.

21. 1 by $\dfrac{a}{b}$.

22. -1 by $\dfrac{x}{y}$.

23. $\dfrac{4ab}{5xy}$ by $\dfrac{xy}{ab}$.

24. $2+\dfrac{4a^2b}{5xy}$ by $\dfrac{2c}{x^2y^2}$.

25. $3+\dfrac{a}{x}$ by $3-\dfrac{a}{x}$.

26. $a-\dfrac{b^2}{a}$ by $a+2b+\dfrac{b^2}{a}$.

27. $\dfrac{a^3+b^3}{a-b}$ by $\dfrac{a+b}{a^3-b^3}$.

28. $a-b+\dfrac{b^2}{a+b}$ by $\dfrac{a}{a^2-b^2}$

29. $\dfrac{x^2-x-30}{x^2+x-6}$ by $\dfrac{x^2+8x+15}{x^2-8x+12}$.

30. $\dfrac{a^2-2ax+x^2}{x+3}$ by $\dfrac{a^2-2ax+x^2}{x-3}$.

31. $\dfrac{x-4}{x+1}$ by $\dfrac{x-2}{x+3}$.

32. $\dfrac{a-b}{c+d}$ by $\dfrac{c-d}{a+b}$.

33. $\dfrac{x-1}{x+7}$ by $\dfrac{x-8}{x+6}$.

Divide:

34. $a^7 - b^7$ by $a - b$.

35. $x^3 + xy + y^2$ by $x + y$.

36. It took a boat's crew that can row in still water 8 miles an hour, 4⅖ hours to row back up a stream a distance that they rowed down in two hours; how fast did the stream flow?

LESSON XXVIII.

Equations of the First Degree.

NOTE. The teacher may help on those numbered in full face type. Then the pupils should do the others.

1. $x - (36 - x) = 4.$ *Ans.* 20.

2. $x + (7 + x) = -17.$ *Ans.* -5.

3. $\frac{2x+1}{2} = \frac{7x+5}{8}.$ *Ans.* 1.

4. $\frac{x+2}{2} = \frac{3x+1}{4}.$

5. $\frac{3x+3}{5} + \frac{2x+1}{3} = \frac{5x-2}{3}.$

6. $\frac{5x-11}{4} - \frac{x-1}{10} = \frac{11x-1}{12}.$

7. $x + \frac{19-2x}{3} = \frac{10+x}{2}.$

8. $x + \frac{x+7}{2} - \frac{2x+7}{3} = 4x - 21.$

9. $\frac{x+3}{2} - \frac{x-2}{3} = \frac{x+2}{2} - \frac{1}{6}.$

10. $\dfrac{4x+3}{10}-\dfrac{12x-5}{5x-29}=\dfrac{2x-1}{5}.$

11. $\dfrac{9x+20}{36}-\dfrac{4x-12}{5x-4}=\dfrac{x}{4}.$

12. $\dfrac{2x-8}{3}-\dfrac{5x-2}{x+1}=\dfrac{4x+7}{6}.$

13. $\dfrac{3x+7}{4}+\dfrac{2x+1}{3x+3}=\dfrac{3x+11}{4}.$

14. $\dfrac{5x+7}{3x+2}=\dfrac{5x-5}{3x-4}.$ *Ans.* 3.

15. $\dfrac{3x}{4x+1}+\dfrac{x}{4x-2}=1.$ *Ans.* 2.

16. $\dfrac{8x+7}{5x+4}=2-\dfrac{2x}{5x+1}.$

17. $\dfrac{1}{x-7}-\dfrac{1}{x-4}=\dfrac{1}{x-1}-\dfrac{1}{x+2}.$ *Ans.* $2\frac{1}{2}$.

NOTE. First simplify the members separately.

18. $\dfrac{1}{x-1}-\dfrac{1}{x-3}=\dfrac{1}{x+2}-\dfrac{1}{x}$ *Ans.* $\frac{2}{3}$.

19. $\dfrac{1}{x-7}-\dfrac{1}{x+7}=\dfrac{1}{x-9}-\dfrac{1}{x+5}.$ *Ans.* 1.

20. $\dfrac{1}{x-10}-\dfrac{1}{x-7}=\dfrac{1}{x-9}-\dfrac{1}{x-6}.$

21. $\dfrac{1}{x+3}-\dfrac{1}{x+1}=\dfrac{1}{x+7}-\dfrac{1}{x+5}.$

22. $\dfrac{x+5}{x+4}-\dfrac{x-6}{x-1}=\dfrac{x-4}{x+5}-\dfrac{x-15}{x-16}.$

23. $\dfrac{x+3}{x+6}-\dfrac{x+6}{x+9}=\dfrac{x+2}{x+5}-\dfrac{x+5}{x+8}.$

24. $$\frac{x-7}{x-9}-\frac{x-9}{x-11}=\frac{x-13}{x-15}-\frac{x-15}{x-17}.$$

25. $$\frac{x+2}{x}-\frac{x+3}{x+1}=\frac{x-6}{x-4}-\frac{x-7}{x-5}.$$

LESSON XXIX.

Problems of the First Degree Solved by the Use of One Letter.

NOTE. If the teacher has to give any help on these problems, it should be on those numbered in the heavier type as 4, 8, etc., only.

1. Divide a string 42 inches long into two parts such that one of them will be three-fourths the length of the other. *Ans.* 24 and 18 inches.

2. The sum of two numbers is 85, and the difference 7; what are the numbers?

3. The sum of two quantities is a, the difference b; what are the quantities? *Ans.* $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

4. A can do a piece of work in 10 days and B in 8; how long will it take them both to do it?

5. A and B can do a piece of work in 5 days and B alone in 9 days; how long will it take A alone to do it? *Ans.* 11¼ days.

6. It takes A twice as long as it does B to do a piece of work, and both together can do it in 4 days; how long would it take each? *Ans.* A in 12 days and B in 6.

7. A can do a piece of work in a days and B in b days; how long would it take both to do it?

8. A workman was engaged to work for thirty days with the agreement that for every day he worked he was to receive $2.00, and for every day he was absent he would forfeit fifty cents; he received $47.50; how many days did he work?

9. A man expends one-fifth his income for board, three-twentieths for lodging, one-tenth for clothes, three-eighths for miscellaneous expenses, and had left $175; what was his income?

10. A man is three times as old as his son, but in 5 years more he will be only 2½ times as old; how old are they each?

11. Two casks contain an equal quantity of oil, but when 5 gallons were drawn from one and 25 from the other, the first contained three times as much as the other; how many gallons were there in each at first?

12. To a certain number add 18, and also from the same number subtract 2; three times the sum equals nine times the difference; what is the number?

13. The difference between two numbers is 6, and if 27 be added to their sum, it will give 29; what are the numbers?

14. The sum of two numbers is 16, and if 12 be added to their difference, the result will be 20; what are the numbers?

15. After paying the fourth and the fifth of a debt, $38.50 is still due; what was the debt?

16. After paying the fifth of a debt, and one-third of the remainder, $48.00 is still due; what was the debt?

17. After paying the half of a debt, one-third of the remainder, and one-fourth of what was then left, and lastly \$7.00, there was still due \$5.00; what was the original debt?

18. After selling a third of his sheep, a fourth of the remainder, and a third of what he then had, a farmer had 20 sheep left; how many had he at first?

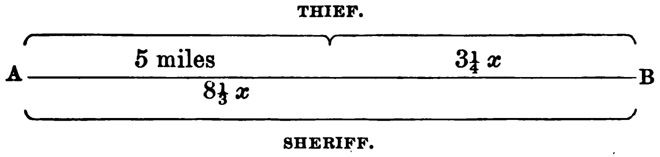
19. The difference between the squares of two consecutive numbers is 25; what are the numbers?

20. The difference between the squares of two consecutive even numbers is 28; what are the numbers?

21. The difference between the squares of two numbers, one of which is 3 larger than the other, is 3; what are the numbers?

22. A thief travelling at the rate of $3\frac{1}{4}$ miles an hour is 5 miles ahead of the sheriff pursuing at the rate of $8\frac{1}{3}$ miles an hour; how long before the former will be overtaken?

Illustrate thus on a scale:



23. An hour after a freight train going at the rate of 18 miles an hour has left the station, it is followed by an express going at the rate of 42 miles an hour; how long before the express will overtake the freight?

24. A steamboat going at the rate of $9\frac{1}{3}$ miles an hour is 7 miles ahead of another going at the rate of $11\frac{5}{6}$ miles per hour; how long before the second will be 7 miles ahead of the first?

25. The minute-hand of a watch goes one minute space per minute, and the hour-hand goes $\frac{1}{12}$ of a minute space per minute; at 4 o'clock, the hour-hand has a lead of 20 spaces; how long before the minute-hand overtakes it?

Ans. $21\frac{9}{11}$ minutes.

26. The hour and minute hands of a watch are together at 12 o'clock; when will they be together next.

Ans. At $5\frac{5}{11}$ minutes past one.

27. The hour and minute hands of a watch are together at 12 o'clock; how long before they will be pointing in opposite directions?

28. At what time between five and six o'clock will the hour and minute hands of a clock be together?

29. At what time between one and two o'clock will the two hands be pointing in opposite directions?

30. A hare is 80 of her own leaps ahead of a greyhound, and takes 3 leaps while the greyhound takes 2; but one of the greyhound's leaps equals two of hers; how many leaps will the hare take before she is caught?

31. A hare is 45 of her leaps ahead of a hound, and is taking five leaps to the hound's 4; but 2 of the hound's leaps equal 3 of the hare's; how many leaps will the hare take before the hound catches her? *Ans.* 225.

32. A hare is 70 of her leaps ahead of a hound, and 280 more leaps will put her in a place of safety. She takes three leaps to the hound's 2, but one of the hound's is equal to 2 of hers; will she be caught?

33. A man is 30 of his own steps behind his son, and takes 3 steps to his son's 5; but one of his steps equals 2 of his son's; how many steps must he take before he overtakes his son?

34. A certain number of persons paid a bill of $1.20; if there had been one more of them, each would have paid 10 cents less; how many were there?

35. A boy bought a number of apples for 18 cents, but had to throw away three of them, making the others cost him half a cent apiece more; how many did he buy?

LESSON XXX.

Equations Using Two or More Letters.

1. $x + y = 16.$
$x - y = 4.$

2. $ax + by = m.$
$ax - by = n.$

3. $x + y = 6.$
$x - y = 20.$

4. $x + y = 5.$
$3x - 2y = 30.$

5. $6x + y = 32.$
$2x - 7y = -4.$

6. $8x - 3y = 5.$
$7x - 2y = 5.$

7. $\dfrac{5x + 2y}{13} = 2.$
$\dfrac{4x - 2y}{5} = 2.$

8. $\dfrac{3x + 5y}{2} = \dfrac{7x + 20}{3}.$
$\dfrac{2x - y}{9} = \dfrac{5y - 20}{5}.$

9. $\dfrac{4x + 7y}{4} = 1.$
$\dfrac{5x - 2y}{23} = 2.$

10. $x + y + z = 9.$
$3x - 2y - z = 8.$
$2x - y + z = 8.$

11. $x - y + 2z = 2.$
$2x - 3y + z = -9.$
$3x + y - 8z = 6.$

12. $x + y = 11.$
$x + z = 13.$
$y + z = 12.$

13. $5x + z = 10.$
$2x + 3y = 8.$
$2y - z = 1.$

14. $2x - 3y + 2z = 9.$
$x - z = -1.$
$6y - 3z = 3.$

15. $(x + 3)(y + 5) = xy + 32.$
$(x + 1)y = xy + 2x - 8.$

16. $\dfrac{x + y}{x - y - 2} = \dfrac{1}{2}.$
$\dfrac{x - y}{x + y + 4} = \dfrac{1}{2}.$

17. $\dfrac{2x - 3y - 3}{3x - y} = \dfrac{1}{3}.$
$\dfrac{x - 3y}{x - 5} = \dfrac{1}{3}.$

18. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{7}{10}.$
$\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{3}{10}.$

Note. Eliminate one of the unknown quantities before clearing of fractions.

19. $\dfrac{3}{x} + \dfrac{5}{y} = 2.$
$\dfrac{6}{x} + \dfrac{7}{y} = 2.$

20. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{13}{12}.$
$\dfrac{2}{x} + \dfrac{1}{y} - \dfrac{1}{z} = \dfrac{13}{12}.$
$\dfrac{3}{x} - \dfrac{4}{y} - \dfrac{2}{z} = 0.$

21. $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{12}.$
$\dfrac{1}{y} - \dfrac{1}{z} = \dfrac{1}{8}.$
$\dfrac{1}{y} + \dfrac{1}{z} = \dfrac{7}{24}.$

22. $x + y + z = 0.$
$3x - y + 5z = 2.$
$2x - y + 4z = 0.$

23. $\dfrac{a}{x} + \dfrac{a}{y} = \dfrac{7a}{12}.$
$\dfrac{a}{x} + \dfrac{a}{z} = \dfrac{a}{2}.$
$\dfrac{a}{y} - \dfrac{a}{z} = \dfrac{a}{12}.$

LESSON XXXI.

Problems.

Note. **Use two letters to express the unknown quantities. Have pupils prove their answers where answers are not given.**

1. The sum of two numbers is 43, and their difference 11; what are the numbers?

2. A boat's crew rowed up a river 11 miles in 2 hours, and down the same river, 10½ miles in 1 hour; how fast can the crew row, and what is the velocity of the stream? *Ans.* Crew rows 8 miles per hour. Stream flows 2½ miles per hour.

3. A steamboat on a river makes two landings 21 miles apart. Going down, her time between the landings is 1 hour and 30 minutes, but going up it is 2 hours and 6 minutes; what is her speed, and what is the velocity of the current? *Ans.* Steamboat 12 miles. Current 2 miles.

4. A farmer sold 20 bushels of wheat and 25 bushels of barley for $32.25, and 50 bushels of wheat and 5 bushels of barley for $43.25; what was the price of each?

5. A farmer sold 3 horses and 4 cows for $340, and at the same price 2 horses and 1 cow for $185; what was the price of each?

6. A number consists of two digits whose sum is 11; if 63 be added to the number the figures will be reversed; what is the number? *Ans.* 29.

7. The sum of the two digits of a number is 12, and if 36 be subtracted from the number, the digits will be reversed; what is the number?

8. The difference between the two digits of a number is 3, the 10's figure being the larger, and if 2 more than five times the sum of the digits be subtracted from the number, the digits will be reversed; what is the number?

9. If 3 be added to the denominator of a certain fraction, it becomes $\frac{1}{3}$; if 2 be added to the numerator, it becomes $\frac{2}{3}$; what is the fraction? *Ans.* $\frac{4}{9}$.

10. If 2 be added to the denominator of a certain fraction, it becomes $\frac{1}{3}$; if 1 be subtracted from the denominator, it becomes $\frac{1}{2}$; what is the fraction?

11. A certain fraction becomes $\frac{1}{2}$ by adding 1 to the numerator, and $\frac{1}{3}$ by adding 1 to the denominator; what is the fraction?

12. A and B together can do a piece of work in 6 hours, but A worked 7 hours and B finished it in 4; how long would it take each alone? *Ans.* A 9 hours, and B 2.

13. A and B can do a piece of work in 2 hours and 24 minutes; and they can also do it if A works 3 hours and B 2; how long would it take each alone?

LESSON XXXII.*

Involution Continued — Application of the General Formulas for the Binomial Theorem.

General Formulas:

$$(a \pm b)^2;\ (a \pm b)^3;\ (a \pm b)^4, \text{ etc.}$$

Expand:

1. $(2a + 3)^2$.

2. $(3 - 3b)^2$.

3. $(2x + 3y)^3$.

4. $(3x - 2y)^4$.

5. $(1 - 2x)^5$.
6. $(3x + 2y)^5$.
7. $(5x - 2y)^3$.
8. $(a^2 - b^2)^4$.
9. $(x^2 - y^2)^3$.
10. $(a^2 + b^3)^5$.
11. $(x^2y - 2)^4$.
12. $(x^3 + y^3)^3$.
13. $(xy - a)^3$.
14. $(ax^2 + bx)^3$.
15. $(a^2x + b^2y)^2$.
16. $(2ax - 2ay)^2$.
17. $\left(\frac{x}{2} + y\right)^3$.
18. $\left(\frac{x}{2} + 2y\right)^2$.
19. $\left(\frac{2x}{3} + \frac{3y}{2}\right)^2$.
20. $(a + (b + c))^2$.
21. $(m + n + r)^2$.
22. $(2x + 3y + z)^2$.
23. $(3a + b + 2)^2$.
24. $(x + 2y + 3z)^2$.
25. $(a + b - c)^2$.
26. $(a - b + c)^2$.
27. $(a - b - c)^2$.
28. $(43)^2 = (40 + 3)^2$.
29. $(56)^2$.
30. $(37)^2$.

Expand (mentally):

31. $(49)^2 = (50 - 1)^2$.
32. $(59)^2$.
33. $(68)^2$.
34. $(37)^2$.
35. $(51)^2$.

LESSON XXXIII.*

Evolution, Square Root — Pure Equations of the Second Degree.

Formula:

$$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)b.$$

Extract the square root of

1. $x^2 + 6x + 9$.
2. $x^2 - 8x + 16$.

3. $4x^2 + 12xy + 9y^2$. **4.** $25x^4 - 20x^2y + 4y^2$.

5. $49a^4 - 42a^2 + 9$.

6. $81x^4 + 126a^2b^2 + 49b^4$.

7. $64x^6 - 160x^3y^2 + 100y^4$.

8. $36a^2x^4 + 36abx^2y + 9b^2y^2$.

9. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

Find roots of the following:

10. $\sqrt{1225}$. **15.** $\sqrt{2}$ (to two decimal places).

11. $\sqrt{5625}$. **16.** $\sqrt{3}$ (to two decimal places).

12. $\sqrt{8649}$. **17.** $\sqrt{5}$ (to two decimal places).

13. $\sqrt{54756}$. **18.** $\sqrt{7}$ (to two decimal places).

14. $\sqrt{15625}$. **19.** $\sqrt{8}$ (to two decimal places).

Find the value of x and y:

20. $x^2 = 144$. *Ans.* ± 12.

21. $x^2 = 81$. **22.** $x^2 + 7 = 56$.

23. $x^2 - 3a^2 = 22a^2$. *Ans.* $\pm 5a$.

24. $x^2 + y^2 = 117$.
$x^2 - y^2 = 45$.

25. $5x^2 - y^2 = 109$. *Ans.* $x = \pm 5$.
$x^2 + y^2 = 41$. $y = \pm 4$.

26. $4x^2 + 2xy + 7 = x^2 + 2xy + 307$.

27. $x^2 = 15$. *Ans.* $x = 3.87 +$.

28. $x^2 = 18225$. **33.** $\sqrt[3]{3ax} = 6a$.

29. $x^2 - 37636 = 0$. **34.** $\sqrt[4]{x} = 2a$.

30. $\sqrt{x} = 5$. **35.** $\sqrt{8x} = 4$.

31. $\sqrt[3]{x} = 7$. **36.** $\sqrt[3]{3x} = 6$.

32. $\sqrt{2x} = 16$. **37.** $\sqrt{5x} = 15$.

38. $\sqrt[3]{\frac{x}{2}} = 2.$ **39.** $\sqrt{\frac{x}{3}} = 12.$

40. Give the six axioms chiefly used in solving equations.

LESSON XXXIV.*

Evolutions, Cube Root — Equations of the Third Degree.

Formula:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$$

Cube root of:

1. $x^3 + 3x^2y + 3xy^2 + y^3.$

2. $a^3 - 18a^2 + 108a - 216.$

3. $8a^3 + 36a^2 + 54a + 27.$

4. $8x^3 - 36bx^2 + 54b^2x - 27b^3.$

5. $125x^3 + 150x^2y + 60xy^2 + 8y^3.$

6. $343 - 294a + 84a^2 - 8a^3.$

Extract:

7. $\sqrt[3]{74088}.$ **10.** $\sqrt[3]{21024576}.$

8. $\sqrt[3]{132651}.$ **11.** $\sqrt[3]{5}$ (to one decimal place).

9. $\sqrt[3]{57912}.$ **12.** $\sqrt[3]{12}$ (to one decimal place).

Find the value of x and y:

13. $x^3 = 343.$ **15.** $x^3 - 13824 = 0.$

14. $x^3 = -729.$ **16.** $x^3 = 512a^6.$

17. $x^3 + y^3 = 35.$ *Ans.* $x = 3.$

$x^3 - y^3 = 19.$ $y = 2.$

18. $x^3 + y^2 = 368.$ $x^3 - y^2 = 318.$

19. $x^3 + xy = 270.$ $x^3 - xy = 162.$

Expand:

20. $(2a - b)^5.$

21. $\left(\frac{a}{2} + 4\right)^4.$

22. $\left(\frac{x}{3} + 3y\right)^3.$

23. $\left(2b + \frac{a}{2}\right)^4.$

24. $(2x^2 - 3y^2)^3.$

25. $(x^2 + y^2)^5.$

LESSON XXXV.*

Roots of Equations of the First and Second Degrees.

What value of x will satisfy the following equations?

1. $x - 5 = 0.$
2. $x + 4 = 0.$
3. $x - 7 = 0.$
4. $x + 1 = 0.$
5. $x - 1 = 0.$
6. $3x - 4 = 0.$
7. $2x - 4a = 0.$
8. $5x + 15 = 0.$

What values of x will satisfy the following?

9. $(x + 1)(x - 2) = 0.$
10. $(x + 3)(x + 5) = 0.$
11. $(x - 8)(3x - 6) = 0.$
12. $(5x + 15)(3x - 9) = 0.$
13. $(x - 4)(x - 3) = 0.$

14. Of what degree are equations 1 to 8? How many roots has each?

15. Of what degree are equations 9 to 13? (Determine by multiplication.) How many roots has each?

What values of x will satisfy the following?

16. $(x+1)(x-2)(x-3)=0$.

17. $(x-2)(x+2)(x-4)=0$.

18. Problems 16 and 17 are equations of what degree? How many roots has each?

Factor the first member of the following equations, and determine the roots, or values of x; but first, when necessary, make the second member 0 by eliminating from it every positive or negative quantity: —

19. $x^2-7x+12=0$. *Ans.* 3, or 4.

20. $x^2-2x-48=0$. *Ans.* 8, or -6.

21. $x^2+x-42=0$.

22. $x^2-5x=6$.

23. $x^2+2x=-1$.

24. $x^2-4x+4=0$.

25. $x^2-18x+80=0$.

26. $x^2-6ax+8a^2=0$.

Factoring; Case XI.

General Formulas:

1. $a^2x^2+(b+c)ax+bc=(ax+b)(ax+c)$.
2. $a^2x^2-(b+c)ax+bc=(ax-b)(ax-c)$.
3. $a^2x^2+(b-c)ax-bc=(ax+b)(ax-c)$.

Factor:

27. $9x^2-24x+15$.

28. $25x^2+15x+2$.

29. $16x^2-12x-10$.

30. $4x^2+6x-18$.

Factor the first member of the following equations, and determine the roots, or values of x: —

31. $9x^2 - 24x + 15 = 0.$ *Ans.* 1, or $1\frac{2}{3}$.

32. $36x^2 - 30x - 6 = 0.$ *Ans.* 1, or $-\frac{1}{6}$.

33. $4x^2 + 40x + 96 = 0.$

34. $25x^2 + 25x - 50 = 0.$

35. $16x^2 - 16x - 32 = 0.$

36. $49x^2 - 35x - 14 = 0.$

37. $b^2x^2 - 3b^2x + 2b^2 = 0.$

For Case XII., see Lesson XXXIX.

Extract:

38. $\sqrt{36x^2 - 60xy + 25y^2}.$

39. $\sqrt[3]{8x^3 + 36x^2 + 54x + 27}.$ *Ans.* $2x + 3.$

40. $\sqrt{x^4 + 4x^3 - 2x^2 - 12x + 9}.$

Ans. $x^2 + 2x - 3.$

LESSON XXXVI.

Affected Quadratic Equations Solved by Completing the Square.

Find the

1. $\sqrt{x^2 + 8x + 16}.$

2. $\sqrt{x^2 - 7x + 12\frac{1}{4}}.$

3. $\sqrt{x^2 + 3x + 2.25}.$

Add a third term to the following binomials that will make them perfect squares, and extract the root: —

4. $x^2 + 10x.$

5. $x^2 + 12x.$

6. $x^2 - 6x.$

7. $x^2 + 5x.$

8. $x^2 - 9x.$

9. $x^2 + 4ax.$

In the same way, add to both members of each of the following equations a quantity that will make the first member a perfect square, and then find the roots of the equation: —

10. $x^2 + 4x = 45.$

11. $x^2 - 14x = -48.$

12. $x^2 + 5x = 24.$

13. $4x^2 - 12x = 40.$

14. $9x^2 + 18x = 72.$

Find the roots of the following equations to two decimal places: —

15. $x^2 + 6x = 42.$

16. $x^2 - 7x = 20.$

Find the roots of the following equations, either by completing the square, or by factoring as in the preceding lesson: —

17. $x^2 + 14x = -49.$ *Ans.* -7 and -7.

18. $x^2 - 11x = 30.$ *Ans.* 5, and 6.

19. $9x^2 - 15x - 14 = 0.$

20. $x^2 - 8x + 15 = 0.$

21. $3x^2 + 6x - 24 = 0.$

22. $2x^2 - 4x = 30.$

23. $25x^2 - 20x - 60 = 0.$

24. $x + 3 = \frac{40}{x}.$

25. $x - 5 = \frac{6}{x}.$

26. $x + 2 = \frac{20}{x + 1}.$

27. $x = \frac{63}{x + 2}$

28. $2x + 5 = \frac{273}{2x - 3}.$

29. $3x - 10 = \frac{28}{3x + 2}.$

30. $\frac{5}{x}+x=6.$ **31.** $\frac{x+2}{x-2}=\frac{2x+7}{x+1}.$

32. $\frac{x+4}{3x-8}=\frac{x+5}{2x-1}.$

LESSON XXXVII.

Problems Producing Affected Quadratic Equations.

NOTE. In a few cases it may be interesting to try to interpret the negative value of x, but as a rule it should be disregarded.

1. A rectangular board is 5 inches longer than it is wide, and contains 234 square inches; what are its dimensions? *Ans.* 13 by 18.

2. Divide the number 17 into two such parts that their product will be 52.

3. There are two numbers whose difference is 9, and their sum multiplied by the greater is 266; what are the numbers? *Ans.* 14 and 5.

4. A field 4 rods longer than wide contains 140 sq. rods; what are the dimensions of the field?

5. Divide the number 19 into two such parts that their product will be 84.

6. A boy bought some oranges for 24 cents; if he had got 2 more for the 24 cents, the oranges would have cost one cent apiece less; how many did he buy?

7. A boy bought some apples for 12 cents; losing 2 of them, it made the others cost him a cent apiece more; how many did he buy? *Ans.* 6.

8. A woman bought some eggs for 40 cents, but losing 4 of them, the rest cost her one-half a cent apiece more; how many did she buy?

9. The sum of the squares of two consecutive numbers is 313; what are the numbers?

NOTE. Compare with Problem 19, Lesson XXIX.

10. The sum of two consecutive even numbers is 580; what are the numbers?

11. Find a number such that if 15 be added to its square the sum will equal 8 times the number.

12. By going 2 miles an hour more, A went 80 miles in 2 hours less than B; at what rate did each travel?

LESSON XXXVIII.

Arithmetical Problems solved by Algebraic Formulas.

NOTE. Already there has been some work of this kind, as in extracting square root, Lesson XXXIII., and cube root, Lesson XXXIV. The following problems will give a little more extended practice of this kind. First work out the general formula, and then do the arithmetical problems by it.

I. A can do a piece of work in a days (or hours), and B can do it in b days (or hours); how long will it take them to do it together?

Solution:

Let x be the time it will take them both; then,

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$$

Reducing this gives, $x = \dfrac{ab}{a+b}$.

1. A can do a piece of work in 10 hours, and B in 8; how long will it take both?

2. A in 7 days and B in 5 days (?)
3. A in 4 days and B in 3 days (?)

II. Percentage. *General Formulas.*

Let c be the percentage, b the base, and r the rate; then, when b and r are known.

1. $c = br$.

To find the equation for r when b and c are known; first transpose[1] the members of the above equation, which gives $br = c$; hence,

2. $r = \frac{c}{b}$.

In the same way, find the formula: —

3. $b = \frac{c}{r}$.

Remember the rate numerically must be expressed in hundredths, as 5 per cent is .05, 18 per cent, .18, etc.

Problems.

1. 9 per cent of 850.
2. 15 per cent of 620.
3. 13 per cent of 2600.
4. What per cent of 950 is 190?
5. What per cent of 640 is 00?
6. What per cent of 8 is 5?
7. What per cent of 9 is 3?
8. 2½ per cent of what number will give 78?
9. 2½ per cent of what number will give 150?
10. 2½ per cent of what number will give 60?
11. 2½ per cent of what number will give 240?

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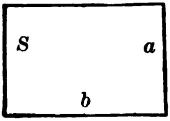
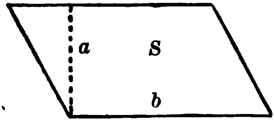
[1] Transpose is here used in its correct sense; for example, if $6 = x$ then by transposing both members, $x = 6$; but there is no change of signs.

III. MENSURATION.

I. *Area of Parallelograms.*





Let S be the surface or area, b the base, and a the altitude.

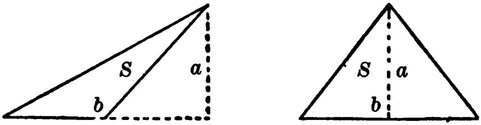
General Formulas.

1. $S = ab$.
2. Find it for a, when b and S are given.
3. Find it for b, when a and S are given.

Problems. (Measurements may be called feet, inches, or any unit.)

1. Base 18, altitude 9; area ?
2. Base 25, altitude 6; area ?
3. Base ? altitude 9, area 810.
4. Base 230; altitude ? area 690.
5. Base ? altitude 27, area 100.

II. *Area of Triangles.*



LETTERS AS ABOVE.

General Formulas.

1. $S = \frac{ab}{2}$.

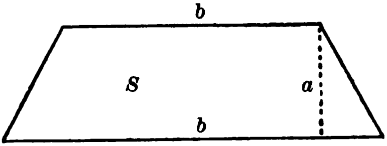
2. Find it for a, when b and S are given.

3. Find it for b, when a and S are given.

Problems.

1. Base 30, altitude 15; area ?
2. Base 32, altitude 20; area ?
3. Base ? altitude 75, area 900.
4. Base 60; altitude ? area 540.
5. Base 84; altitude ? area 1000.

III. *Area of Trapezoids.*



LETTERS AS BEFORE.

General Formulas.

1. $S = \frac{a(b + b')}{2}$.
2. Find it for a when b, b', and S are given.

Problems.

1. Bases 20 and 30, altitude 12; area ?
2. Bases 15 and 35; altitude ? area 500.
3. Bases 9 and 21, altitude 30; area ?

IV. *The Circumference of a Circle.*

Let c be the circumference, R the radius, $2R$ the diameter, and $\pi = 3.1416$, the ratio of the circumference.

General Formulas.

1. $c = 2\pi R$. 2. Find it for $2R$ when c is given.

Problems.

1. Diameter 5; circumference ?
2. Diameter 10; circumference ?

3. Diameter ? circumference 21.9912.
4. Diameter 200; circumference ?
5. Diameter ? circumference 200.
6. Diameter ? circumference 500.

V. *Area of Circle.*

The area of a circle is equal to that of a triangle whose base equals the circumference, and altitude the radius; hence the general formula:—

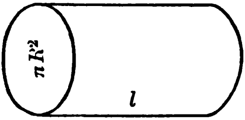
1. $S = 2\pi R \times \frac{R}{2}, = \pi R^2.$
2. Find it for R when s is given.

Problems.

1. Radius 8; area ?
2. Radius 10; area ?
3. Radius ? area 28.2744.
4. Radius ? area 78.54.

VI. *Volume of a Cylinder.*

Let v be the volume, πR^2 the area of one end, and l the length.



General Formulas.

1. $v = l\pi R^2.$
2. Find it for l when v and R are given.

Problems. (Dimensions in inches or feet.)

1. Radius 8, length 15; volume ?
2. Radius 10, length 24; volume ?

3. Radius 6; length ? volume 1500.
4. Radius 3, length 12; volume ?

VII. *Surface of Cylinder.*

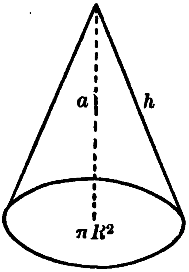
General Formula.

1. $S = 2\pi R^2 + 2\pi Rl = 2\pi R(R + l)$.

Problems.

1. Radius 5, length 20; surface ?
2. Radius 3, length 9; surface ?
3. Radius 1, length 10; surface ?

VIII. *Volume of a Cone.*



Let πR be the area of base and a the altitude, v the volume.

General Formula. 1. $v = \dfrac{a\pi R^2}{3}$.

Problems.

1. Radius of base 4, altitude 9; volume ?
2. Radius of base 5, altitude 24; volume ?

IX. *Surface of a Cone.*

Let h be the slant height.

General Formula. (Curved surface.)

1. $S = h\pi R$.

Total Surface S'.

2. $S' = h\pi R + \pi R^2 = \pi R\,(h + R)$.

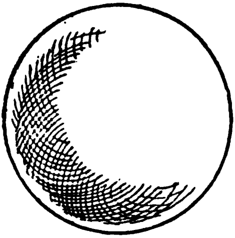
Problems.

1. Radius 3, slant height 8; curved surface ?
2. Radius 5, slant height 10; total surface ?

X. *Surface of a Sphere.*

General Formula.

1. $S = 4\,\pi R^2$.



Problems.

1. Radius 6; surface ?
2. Radius 8; surface ?
3. Radius 5; surface ?
4. Radius 10; surface ?

XI. *Volume of a Sphere.*

General Formula.

1. $v = 4\,\pi R^2 \times \dfrac{R}{3} = \dfrac{4\,\pi R^3}{3}$.

Problems.

1. Radius 2; volume ?
2. Radius 5; volume ?
3. Radius 10; volume ?
4. Find the formula for R when the volume is given.

Problems.

1. Volume 37.6992; radius ?
2. Volume 50.2656; radius ?

LESSON XXXIX.

Factoring. — Case XII. — Permanences and Changes in Signs.

General Formulas:

$$(ax + b)(x + c) = ax^2 + (ac + b)x + bc.$$
$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

Combine by inspection:

1. $(2x + 3)(x + 2)$.
2. $(3x + 5)(x + 1)$.
3. $(2x - 5)(x - 1)$.
4. $(5x - 6)(x - 7)$.
5. $(4x + 7)(x - 2)$.
6. $(3x - 5)(x + 2)$.
7. $(5x + 8)(x + 1)$.
8. $(7x - 2)(x + 3)$.

Factor:

9. $3x^2 + 5x + 2$.
10. $2x^2 + 3x + 1$.
11. $2x^2 + 11x + 12$.
12. $3x^2 + 16x + 5$.
13. $3x^2 - 9x + 6$.
14. $4x^2 - 9x - 9$.
15. $5x^2 - 33x - 14$.
16. $4x^2 - 3x - 27$.
17. $6x^2 + 13x + 7$.
18. $7x^2 + 5x - 18$.

Find the roots of x in the following equations:

19. $5x^2 - 2x - 16 = 0$.
20. $3x^2 - x - 14 = 0$.
21. $4x^2 + 11x = 3$.
22. $7x^2 + 3x = 22$.
23. $2x^2 + x = 10$.
24. $3x^2 - x - 14 = 0$.
25. $5x^2 + 2x - 16 = 0$.
26. $7x^2 - 3x - 4 = 0$.

Multiply by inspection:

27. $(2x + 3)(3x + 2)$.
28. $(2x + 5)(4x + 3)$.
29. $(3x - 7)(5x - 8)$.
30. $(5x - 9)(2x - 1)$.
31. $(3x - 1)(2x - 1)$.
32. $(2x - 1)(5x + 2)$.

33. $(3x-2)(5x+2)$. **35.** $(3x-3)(5x+4)$.

34. $(5x+2)(3x-3)$. **36.** $(5x-1)(3x-1)$.

37. $(3x-2)(7x-2)$.

Factor:

38. $6x^2+x-2$. **43.** $14x^2+3x-27$.

39. $10x^2-7x-12$. **44.** $15x^2-8x-12$.

40. $8x^2+10x-3$. **45.** $10x^2-19x+7$.

41. $6x^2+x-35$. **46.** $6x^2-x-12$.

42. $8x^2+2x-15$. **47.** $15x^2-16x+4$.

48. $6x^2-19x+15$.

49. The length and breadth of a rectangular field are to each other in the ratio of 5 to 3. If the field was 7 rods longer and 2 rods wider, it would contain 242 square rods; find the length and breadth of the field.

Note. Let $5x$ and $3x$ equal the length and breadth respectively.

50. The sides of a rectangular field are in the ratio of 3 to 2. If the length was 5 rods less, and the width 3 rods more, the area would be 361 square rods; find the dimensions of the field.

51. The length of a mat is to its width in the ratio of 4 to 3. If it were 3 feet shorter and 2 feet narrower, it would contain seven square yards; find the dimensions of the mat.

52. $\dfrac{60}{7x-5}+\dfrac{35}{2x-3}=7$.

53. A man bought a number of pounds of coffee. The number of pounds was to the number of cents he paid per pound as 5 to 7. If he had bought 5 pounds more at a price 5 cents per pound less, his coffee would have cost him $9.00; how much did he buy, and what price did he pay?

54. $\frac{x+2}{x-2} - \frac{x-2}{x+2} = \frac{5}{6}.$ **55.** $\frac{x+4}{x-4} - \frac{x-4}{x+4} = \frac{10}{3}.$

56. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

57. $\frac{3x-2}{2x-5} - \frac{2x-5}{3x-2} = \frac{8}{3}.$

58. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}.$ **59.** $5x^2 - 55 = -14x.$

60. $8x^2 + x = 30.$ **61.** $4x^2 + 17x = 15.$

62. A train goes 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the time would have been 2 hours less; what is the rate of the train?

63. A man travels 108 miles, and finds that he could have made the journey in 4½ hours less if he had travelled 2 miles an hour faster; at what rate did he travel?

64. Two rectangles contain the same area, 480 square yards. One is 10 yards the longer, and the other is 4 yards the wider; find their sides.

65. A cistern can be filled by two pipes running together in 22½ minutes; the larger pipe can fill the cistern in 24 minutes less than the smaller one; find the time in which each would fill the cistern.

MISCELLANEOUS EXAMPLES.

Multiply:

1. $a^2 - 3a + 2$ by $a - 3$.
2. $9a^2 + 6ab + 4b$ by $3a - 2b$.
3. $x^2 - 3x + 2$ by $x^2 + 3x - 2$.
4. $a + b + c$ by $a - b - c$.
5. $x^2 - y^2 - 2yz - z^2$ by $x^2 - y^2 + 2yz - z^2$.
6. $\frac{a+b}{a} - \frac{a+b}{b}$ by $\frac{a+b}{a} + \frac{a-b}{b}$.
7. $\frac{x-3}{x-2} - \frac{x-5}{x+2}$ by $\frac{x-1}{x+2} - \frac{x-7}{x-2}$.

Divide:

8. $3a^4 - a^3 + 2a^2 + a + 1$ by $a^2 - a + 1$.
9. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
10. $27a^3 - 8b^3$ by $9a^2 + 6ab + 4b^2$.
11. $32x^5 + y^5$ by $16x^4 - 8x^3y + 4x^2y^2 - 2xy^3 + y^4$.
12. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$ by $a + b$.

Multiply together:

13. $\frac{ab}{xy}$, $\frac{yz}{bc}$, and $\frac{cd}{zv}$.

14. $\frac{a^2bc}{x^2y^2}$, $\frac{ac^3}{xy^3}$, and $\frac{x^4y^5}{a^4bc^3}$.

15. Two gentlemen saw a sum of money lying on a table. One said, " I have five times as much money as there is on that table." The other replied, " I have ten

times as much." Together they had \$105; how much money was there on the table?

16. A company of 90 persons consists of men, women, and children. There are 4 more men than women, and ten more children than adults; how many are there of each?

17. A clerk was five years in the same house. In the first three years he spent \$400 a year, but in each of the following years he spent \$100 more than in the preceding year. At the end of the five years he had saved \$1,700; what was his salary?

18. $\frac{x}{2}+\frac{3x}{5}-\frac{5x}{6}+\frac{7x}{12}=51$, to find x.

19. $\frac{3x}{4}-\frac{5x}{6}+\frac{x}{12}+\frac{2x}{3}=64$, to find x.

20. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}+\frac{x}{d}=m$, to find x.

21. $x+\frac{x}{4}+\frac{x}{5}-\frac{x}{6}=2x-43$, to find x.

22. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find x.

23. A fish was caught whose tail weighed 9 lbs.; its head weighed as much as its tail and half its body, and its body weighed as much as its head and tail together; what was the weight of the fish?

24. A man engaged a workman for 50 days; for each day he worked he received \$1.50, and for each day he was idle he paid 50 cents for board. At the end of the 50 days he received \$45; how many days did he work?

25. An estate of \$7,500 is to be divided between a widow, two sons, and three daughters, so that each son

shall receive twice as much as each daughter, and the widow \$500 more than all the children together; how much did each receive?

26. A pole is one-fourth in the mud, one-third in the water, and 10 feet above the water; what is the length of the pole?

27. A man going to market put a certain sum of money in his pocket. One-third of it he paid for sugar, one-fourth for coffee, one-sixth for tea, one-seventh for rice, and the remainder, amounting to \$4.50, for salt; how much did he put in his pocket?

28. In a mixture of copper, tin, and lead, 16 lbs. less than one-half of the mixture is copper; 12 lbs. less than one-third is tin; and 4 lbs. more than one-fourth is lead; what quantity is there of each in the mixture?

Combine at sight:

29. $(x + 7)(x - 6)$.

30. $(x + 12)(x - 9)$.

31. $(x - 7a)(x - 2a)$.

32. $(x + 9)(x + 1)$.

33. $(2x - 7)(2x - 3)$.

34. $(3x - 4)(3x + 5)$.

35. $(7ab - 3xy)(7ab + 3xy)$.

Factor:

36. $x^2 - 12x + 11$.

37. $x^2 - x - 56$.

38. $x^2 + 2x - 80$.

39. $x^4 - 16y^4$.

40. $4x^2 - 12x + 5$.

41. $16x^4 - 40x^2 + 9$.

42. $25x^4 + 5x^2 - 20$.

43. $27x^3 - 8y^6$.

44. Expand $(5a - 3d)^3$.

45. Expand $(3\,a^2c - 2\,bd)^4$.

46. Expand $\left(a - \frac{2\,b}{3}\right)^8$.

47. Expand $\left(\frac{c}{2\,b} + \frac{2\,b}{c}\right)^4$.

48. Expand $\left(\frac{2\,d}{3} - \frac{3\,b}{2}\right)^8$.

Multiply together:

49. $\frac{a^2 - 4\,b^2}{ac^2 + 2\,a^2}$ and $\frac{2\,a}{a - 2\,b}$.

50. $\frac{x^2 + 5\,x + 6}{x^2 - 1}$ and $\frac{x^2 - 2\,x - 3}{x^2 - 9}$.

51. $\frac{x^2 - x - 20}{x^2 - 25}$, $\frac{x^2 - x - 2}{x^2 + 2\,x - 8}$, and $\frac{x^2 + 5\,x}{x + 1}$.

52. $\frac{x^2 - 64}{x^2 + 24\,x + 128}$ by $\frac{x^2 + 12\,x - 64}{x^3 - 64}$, and divide the product by $\frac{x^2 - 16\,x + 64}{x^2 + 4\,x + 16}$.

53. $\frac{4\,x^2 - 16\,x + 15}{2\,x^2 + 3\,x + 1}$ by $\frac{x^2 - 6\,x - 7}{2\,x^2 - 17\,x + 21}$, and divide the product by $\frac{4\,x^2 - 20\,x + 25}{4\,x^2 - 1}$.

54. G. C. D. of $x^3 - x^2 - 41\,x + 105$, and $x^4 - 2\,x^3 - 19\,x^2 + 8\,x + 60$.

NOTE. Divide the greater by the less, and factor the remainder and the divisor.

55. G. C. D. of $x^5 - 5\,x^4y + 10\,x^3y^2 - 10\,x^2y^3 + 5\,xy^4 - y^5$, and $x^3 - x^2y - xy^2 + y^3$.

56. G. C. D. of $x^4 - x^3 + 8\,x - 24$, and $x^3 + 4\,x^2 - 8\,x - 8$.

57. G. C. D. of $x^3 + 3x^2 - 8x - 24$, and $x^3 + 3x^2 - 3x - 9$.

58. G. C. D. of $x^4 - 3x^3 + 2x^2 + x - 1$, and $x^3 - x^2 - 2x + 2$.

59. Simplify $\dfrac{a^2 - b^2}{a^2 - 3ab + 2b^2} \times \dfrac{ab - 2b^2}{a^2 + ab}$

$$\times \frac{a^2 - ab}{a^2 - 2ab + b^2}.$$

60. $x^2 - 15x + 54$; find roots of the equation.

61. $4x^2 + 12x = 40$; to find the roots.

62. $(x + 5)^2 + 8(x + 5) = 180$; find the roots.

63. $x^2 - 4ax = 5a^2$; find the roots.

64. A number consists of two digits whose sum is 13; if 45 be added to the number, the figures will be reversed; what is the number?

65. A man paid out one-fourth of the money he had, and afterwards received $3. Again he paid out a third of what he then had, and afterwards received $2. Lastly he paid out one-eighth of what he then had, and found he had $14 left; how much had he at first?

66. A farmer, wishing to sell his flock of sheep, sells half of the whole number and half a sheep over. Next he sells half of what he had left and half a sheep over. This he does a third and fourth time, disposing at each sale of half the sheep left on his hands and half a sheep over. A fifth sale of eight sheep disposed of the entire flock; how many sheep had he at first?

67. If two apples cost one cent and three pears two cents, and I buy 100 apples and pears for 56 cents, how many of each do I buy?

68. $$\begin{aligned} 5x + \frac{y}{2} &= 24 \\ \frac{3x}{2} - \frac{y}{4} &= 4 \end{aligned}$$

69. $$\begin{aligned} \frac{x+3}{4} - y &= -2 \\ \frac{x-3}{4} + 4y &= 21\tfrac{1}{2}. \end{aligned}$$

70. A owes \$1200. and B owes \$2500. but neither has enough to pay his debts. Said A to B, "Lend me the eighth of what you have and I can pay my debt."—"Lend me," replied B, "the ninth part of what you have, and I can pay mine." How much money had each?

71. A man being asked the ages of himself and his son, replied: "Six years ago I was three times as old as he, but three years hence I shall be only 2½ times as old as he." What is the age of each?

72. At what time between 10 o'clock and 11 o'clock will the minute and hour hand of a clock be together.

73. 34 lbs. of zinc loses 5 lbs. when immersed in water, and 17 lbs. of tin loses 2 lbs. in water. A composition of zinc and tin weighing 136 lbs. loses 19 lbs. in water. How much of each does the composition contain?

74. A certain fraction becomes ½ when 4 is added to the numerator, and ¼ when 2 is subtracted from the denominator; what is the fraction?

75. A and B together buy 150 acres of land; if A had bought twice as much and B only three-fourths as much, they would have together bought 50 acres; how many acres did each buy?

Interpret negative result.

76. Find the square root of $4x^2 + 20xy + 12x + 25y^2 + 30y + 9$.

77. The cube root of $8\,a^6b^3 - 36\,a^5b^4 + 54\,a^4b^5 - 27\,a^3b^6$.

78. The square root of $9\,x^4 - 24\,x^3 + 46\,x^2 - 40\,x + 25$.

79. The square root of $a^4x^6 + 2\,a^3x^5 + a^2x^4 - 4\,a^3x^4 - 4\,a^2x^3 + 4\,a^2x^2$.

80. The cube root of $a^3x^6 - 3\,a^5x^5 + 3\,a^7x^4 - a^9x^3$.

81. The square root of $4\,x^6 - 12\,x^5 + 13\,x^4 - 6\,x^3 + x^2$.

82. The square root of $25\,x^4 - 30\,x^3y - 11\,x^2y^2 + 12\,xy^3 + 4y^4$.

83. The cube root of $8\,a^3x^3 - 36\,a^2bx^2y + 54\,ab^2xy^2 - 27\,b^3y^3$.

Simplify the following:

84. $\dfrac{5}{x^2 - 5\,x + 6} + \dfrac{5}{x^2 + x - 6}$.

85. $\dfrac{x + 2}{x - 5} - \dfrac{x - 3}{x + 5} + \dfrac{x + 8}{x + 2}$.

86. $\dfrac{2\,x^2}{x^2 - y^2} - \dfrac{2\,x}{x + y} + \dfrac{4\,y}{x - y} - \dfrac{y}{x + y}$.

87. $\dfrac{1}{2\,a - 8\,x} - \dfrac{a}{3\,a^2 - 48\,x^2} + \dfrac{1}{2\,a + 8\,x}$.

88. $\dfrac{3}{8\,(a - x)} + \dfrac{1}{4\,(a + x)} - \dfrac{a - x}{4\,(a^2 - x^2)}$.

89. $\dfrac{24\,x}{9 - 12\,x + 4\,x^2} - \dfrac{3 + 2\,x}{3 - 2\,x} + \dfrac{3 - 2\,x}{3 + 2\,x}$.

Find the roots of the following equations:

90. $5\,x^2 - 75\,x + 250 = 0$.

91. $3\,x^2 + 2\,x - 8 = 0$.

92. $2x^2 - 7ax + 3a^2 = 0.$

93. $4x^2 - 4x - 48 = 0.$

94. $9x^2 + 6x - 63 = 0.$

95. $a^2x^2 - 2ax - 120 = 0.$

96. $4x^2 - 4ax - 15a^2 = 0.$

97. $\frac{2x - 4}{x + 1} = \frac{x + 5}{x + 13}.$

98. $\frac{5x - 7}{x + 1} = \frac{2x + 6}{x + 3} + 1.$

99. $\frac{5x + 9}{x + 5} - \frac{3x + 3}{x + 3} = 1.$

100. $\frac{7x + 4}{x + 1} - \frac{x + 2}{x - 1} = 2.$

101. $\frac{3x - 4}{2x - 4} = \frac{6x}{x + 8}.$

102. $\frac{3x - 1}{4x + 7} = 1 - \frac{6}{x + 7}.$

103. $\frac{x + 3}{2x - 7} - \frac{2x - 1}{x - 3} = 0.$

104. $x + y = 28.$
$xy = 187.$

105. $x^2 + y^2 = 58.$
$xy = 21.$

106. $x^2 - y^2 = -40.$
$x - y = -2.$

107. $9x^2 - 4y^2 = 864.$
$3x + 2y = 36.$

108. $8x^2 + 18y^2 = 272.$
$xy = 10.$

109. A man sold a number of sheep for $168. If he had sold 14 more for the same sum he would have received one dollar less for each sheep; how many sheep did he sell?

Interpret the negative answer.

110. Find a number such that four times its square plus nine times the number itself will equal $7\frac{7}{9}$.

Interpret the negative result.

111. A rectangular field is 5 rods longer than it is wide, and its area is 204 sq. rods; what is the length and width of the field?

112. A man divided $2.40 among a number of boys. If there had been three boys more each would have received 18 cents less; how many boys were there?

ANSWERS TO PROBLEMS.

LESSON IX. PAGE 16.

1. $7x$: 2. $-3ab$: 3. $588a^2$: 4. 0: 5. 1st from 2d, $4b^2 + 4ab - a^2$; 2d from 1st, $a^2 - 4ab - 4b^2$:

6. 1st from 2d, $8b^3 - 24ab^2 + 24a^2b - 8a^3$; 2d from 1st, $8a^3 - 24a^2b + 24ab^2 - 8b^3$: 7. a^5b^3; $2x^4y^4$; a^3b^4c, $25a^4b^2c^2$: 8. $15a^5b + 21a^4b^2 + 15a^3b^3 + 6a^2b^4$:

9. $2ax^3 + 4ax^2 + 2ax$: 10. $2a^2 + ab + 3b^2$:

11. $2 + 3ax + 4a^2x^2 + 5a^3x^3$.

LESSON X. PAGE 17.

7. $8ax^2$: 5. $-25a^3x^2y$: 6. $-5ax^2y^2$: 7. $14a^7x^2y$:

8. $-8a^4b^3c$: 9. $2a^2b - a^3$: 10. $9x^3 + 6x^2 - 12x$:

11. $6ac^2 - 12a^2c$: 12. $7x + 9$: 13. $3 + 2y^2 - 8x^2$:

14. $-2a^6x - 6a^5bx - 14a^3x$:

15. $35a^3bc - 15a^2b^2c + 10abc^3 - 15ab^3c^2$:

16. $24x^4y^2 + 14x^3y^3 + 28x^2y^4$: 17. $a^4bc - ab^4c + abc^4$:

18. $s^4 + r^4 - m^4 - n^4$: 19. $a - a^2 + a^3 + a^4 - a^5$:

20. $14x^2 + 6x^3 - 12x^4 + 6x^5$: 21. $-a$; $-5a$; $7x$: 12; $6x$; $-5ab$; $-27xy$; $4ax$; $b - a$; $x - 4$:

22. -7; $5a$; $12xy$; $-6xy$; $4x$; $-4x$; $3ab$; $-12a^2x$; $6a - 2b$; $3y - 2x$; $c + b - a$.

Lesson XI. Page 19.

10. $a^3 + 3a^2b + 3ab^2 + b^3$:
11. $a^4 - 4a^3b^2 + 6a^2b^3 - 4ab^3 + b^4$:
12. $a^4 - 12a^3 + 54a^2 - 108a + 81$:
13. $a^2 + 2ab + b^2 - c^2$: **14.** $a^2 - 2ab + b^2 - c^2$:
15. $2x^3 - 2x^2 - 17x + 21$: **16.** $3x^4 - 3x^3 - 16x^2 - 8$:
17. $a^3 + ab^2 + a^2b + b^3$:
18. $3x^4 - 18x^3 - 4x^2 - 29x + 30$: **19.** $x^2 - 49$:
20. $x^2 - c^2$: **21.** $3a^4 - 7a^2d + 2ac - 5a + 3a^3c - 7acd$
$+ 2c^2 - 5c$: **22.** $a^3 - 3a^2b + 3ab^2 - b^3$:
23. $x^2 + 4x + 4$: **24.** $x^3 - 15x^2 + 75x - 125$:
25. $x^4 - 2b^2x^2 + b^4$: **26.** $x^6 - a^6$: **29.** $6a^2x$:
30. $a^3 - ab^2$: **31.** $25 - a^2$.

Lesson XII. Page 20.

7. $2d$: **8.** $4x$: **9.** -6: **10.** d: **11.** $a - b$: **12.** $x - a$:
13. $2 - a$: **14.** $1 - 2ab$: **15.** $1 - 3a$:
16. $-3b^4 + 2b^2 - 1$: **17.** $2ab^3c^2 - b^2c$: **18.** $a^2 + 2$:
19. $b - a - c$: **20.** $4a - 7$: **21.** $ab^3c + 4b^4c - 3b^3c^2$:
22. $a^2 - 2ab + b^2$: **23.** $a^2 - ab + ac - ad$: **24.** $b - a$:
25. $x^2 - 2ax + a^2$: **26.** $a - b + c$.

Lesson XIII. Page 22.

11. $a + 2$: **12.** $x - 2$: **13.** $2x - 6$: **14.** $x^2 + 6x - 7$:
15. $a - 2x$: **16.** $3a^2 - 2a + 3$: **17.** $x^2 - 5x + 2$:
18. $2ab + b - c$.

Lesson XIV. Page 23.

4. $a^2 + ab + b^2$: **5.** $a^2 - ab + b^2$: **6.** $a^2 + b^2$:
7. $a^3 + ab^2 - a^2b - b^3$; $a^3 + ab^2 + a^2b + b^3$:
8. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$:

9. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$: 10. $x^4 - x^2y^2 + y^4$: 11. $a^2 - ab + b^2$: 12. $a^4 + a^2b^2 + b^4$.

Lesson XV. Page 25.

25. $a^3 + 6a^2 + 12a + 8$: 26. $b^4 - 12b^3 + 54b^2 - 108b + 8$: 27. $x^3 - 9x^2 + 27x - 27$: 29. $c^2 + 10c + 25$: 30. $c^2 - 12c + 36$: 34. $x^2 + x - 6$: 35. $a^2 - 9$: 36. $x - 3$: 37. $x - 4$.

Lesson XVI. Page 26.

27. $a^3(x + y - 1)$: 29. $5x^2y^2(x + 2y - 1)$: 36. $x^2 - 1$: 37. $x^2 - 4a^2$: 38. $4a^2x^2 - 25b^2y^2$: 41. $(c + d)(c - d)$: 42. $(m^2 + n)(m^2 - n)$: 43. $(2a + 3b)(2a - 3b)$: 48. $(2a - 1)(2a + 1)$: 49. $(1 + 3b)(1 - 3b)$: 51. $a(x + y)(x - y)$: 53. $(x^2 + y^2)(x + y)(x - y)$: 55. $(a + 5)(a - 5) = a^2 - 25$: 56. $2x^2 + 19x + 42$: 57. $m^2 - 36$.

Lesson XX. Page 35.

11. $x - y$: 12. $a + c$: 13. $m + 4$: 14. $6 + a$: 15. $3 + a$.

Lesson XXII. Page 39.

11. $\dfrac{x + y}{x - y}$: 12. $\dfrac{a + b}{a - b}$: 13. $\dfrac{a^2 - ab + b^2}{a - b}$: 15. $\dfrac{x + 3}{x + 5}$.

Lesson XXIII. Page 41.

9. $x^2 - 9$: 10. $a^2b^2 - 4a^2$: 11. $6ab(a^2 - b^2)$: 12. $a^4 - a^2b^2 - a^3b + ab^3$: 13. $a^4 - a^2b + a^3b - ab^3$: 16. $ax^3 + 2ax^2 - 5ax - 6a$.

Lesson XXIV. Page 43.

5. $\dfrac{5x}{6}$: 6. $\dfrac{bx + ax}{ab}$: 7. $\dfrac{7a + 23}{12}$: 9. $\dfrac{(a + b)^2}{ab}$:

13. $\frac{2(a^2+b^2)}{a^2-b^2}$: 14. $\frac{2x^2+8}{x^2-4}$: 30. $\frac{a^2+b^2}{a^2-b^2}$: 33. 0:
34. $\frac{2y}{(x+y)(x+z)(y+z)}$.

Lesson XXV. Page 46.

5. $\frac{4a^2+3ab}{a+b}$: 6. $\frac{2ab}{a+b}$: 9. $\frac{x^2}{x+5}$: 10. $-\frac{2b^2}{a-b}$:
13. $x+9+\frac{17}{x}$: 16. $3x^2-x+2+\frac{7}{x-2}$: 22. 1:
23. $\frac{2x^2}{x^2-9}$: 25. $\frac{a}{a^2+3a+2}$: 26. $\frac{(a-b)^2}{ab}$.

Lesson XXVI. Page 48.

9. $2ab$: 10. $\frac{3ax}{2}(x-y)$: 11. $a^2-2ab+b^2$:
12. $\frac{x^2-25}{x-3}$: 22. $\frac{b}{a^2-b^2}$: 23. $\frac{a}{b^2}$: 24. $\frac{a-b}{c(a+b)}$.

Lesson XXVII. Page 50.

1. $\frac{a^2}{b^2}$: 2. 1: 4. $\frac{y}{b}$: 6. $\frac{x}{5a}$: 7. $\frac{a-b}{x+2}$:
11. $\frac{a^2-x^2}{a^2}=1-\frac{x^2}{a^2}$: 12. $a-b$: 13. $\frac{x^2-4x-5}{x^2+2x-24}$:
24. $\frac{5x^2y^2+2a^2bxy}{5c}$.

Lesson XXXII. Page 61.

4. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$:
5. $1-10x+40x^2-80x^3+80x^4-32x^5$:
6. $243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4$ $+32y^5$: 7. $125x^3-150x^2y+60xy^2+8y^3$:
11. $x^8y^4-8x^6y^3+24x^4y^2-48x^2y+16$:

17. $\frac{x^3}{8} + \frac{3x^2y}{4} + \frac{3xy^2}{2} + y^3$: 18. $\frac{x^2}{4} + 2xy + 4y^2$:
19. $\frac{4x^2}{9} + 2xy + \frac{9y^2}{4}$: 20. $a^2 + 2ab + 2ac + b^2 + 2bc$
$+ c^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$:
22. $4x^2 + 9y^2 + z^2 + 12xy + 4xz + 6yz$.

LESSON XXXIII. PAGE 62.

2. $x - 4$: 3. $2x + 3y$: 4. $5x^2 + 2y$: 5. $7a^2 - 3$:
6. $9x^2 + 7b^2$: 7. $8x^3 - 10y^2$.

LESSON XXXIV. PAGE 63.

1. $x + y$: 2. $a - 6$: 3. $2a + 3$.

LESSON XXXV. PAGE 65.

1. $x = 5$: 2. $x = -4$: 3. $x = 7$: 6. $x = \frac{3}{4}$: 7. $x = 2a$:
9. $x = 2$, or -1: 10. $x = -3$, or -5: 16. $x = -1$,
$+2$, or $+3$: 27. $(3x - 3)(3x - 5)$:
28. $(5x + 2)(5x + 1)$.

ANSWERS

TO

MISCELLANEOUS EXAMPLES.

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1. $a^3 - 6a^2 + 11a - 6$: 2. $27a^3 - 8b^3$:
3. $x^4 - 9x^2 + 12x - 4$: 4. $a^2 - b^2 + 2bc - c^2$:
5. $x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2$:
6. $\frac{b^4 + 2ab^3 + 2a^3b - a^4}{a^2b^2}$: 7. $\frac{4(3x^2 + 16x - 64)}{x^4 - 8x^2 + 16}$:
8. $3a^2 + 2a + 1$: 9. $a^2 - ab - ac + b^2 - bc + c^2$:
10. $3a - 2b$: 11. $2x + y$: 12. $a^4 + a^2b^2 + b^4$: 13. $\frac{ad}{xv}$:
14. $\frac{cx}{a}$: 15. \$7.00: 16. 18 women, 22 men, and 50 children:
17. \$8.00: 18. 60: 19. 96; 20. $\frac{abcdm}{abc + abd + acd + bcd}$:
21. 60: 22. $\frac{3b}{3a - 2b}$: 23. 72 lbs.: 24. 35 days:
25. Widow 4000; each son 1000; each daughter 500:
26. 24 ft.: 27. \$42.00: 28. 128 lbs. of copper; 84 lbs. of tin; and 76 lbs. of lead: 29. $x^2 + x - 42$:
30. $x^2 + 3x - 108$: 31. $x^2 - 9ax + 14a^2$:
32. $x^2 + 10x + 9$: 33. $4x^2 - 20x + 21$:
34. $9x^2 + 3x - 20$: 35. $49a^2b^2 - 9x^2y^2$:
36. $(x - 11)(x - 1)$: 37. $(x - 8)(x + 7)$:
38. $(x + 10)(x - 8)$: 39. $(x^2 + 4y^2)(x + 2y)(x - 2y)$:

40. $(2x-1)(2x-5)$:
41. $(2x+3)(2x-3)(2x+1)(2x-1)$:
42. $(5x^2-4)(5x^2+5)$:
43. $(3x-2y^2)(9x^2+6xy^2+4y^4)$:
44. $125a^3-225a^2d+135ad^2-27d^3$:
45. $81a^8c^4-216a^6c^3bd+216a^4c^2b^2d^2-96a^2cb^3d^3$
$+16b^4d^4$: **46.** $a^3-2a^2b+\frac{4ab^2}{3}-\frac{8b^3}{27}$:
47. $\frac{c^4}{16b^4}+\frac{c^2}{b^2}+6+\frac{16b^2}{c^2}+\frac{16b^4}{c^4}$:
48. $\frac{8a^3}{27}-2a^2b+3ab^2-\frac{27b^3}{8}$: **49.** $\frac{2(a+2b)}{c^2+2a}$:
50. $\frac{x+2}{x-1}$: **51.** x: **52.** $\frac{1}{x-8}$: **53.** $\frac{2x-1}{2x-5}$: **54.** $x-5$:
55. $x^2-2xy+y^2$: **56.** x^2-2x+4: **57.** $x+3$:
58. $x-1$: **59.** $\frac{b}{a-b}$: **60.** $x=9$, and 6:
61. $x=2$, and -5: **62.** $x=5$, and -23:
63. $x=a$, and $-5a$: **64.** 49: **65.** \$24.00: **66.** 143 sheep:
67. 36 pears, and 64 apples: **68.** $x=4$, $y=8$:
69. $x=9$, $y=5$: **70.** A, \$900; B, 2400:
71. Father 36, son 15: **72.** $54\frac{6}{11}$ minutes past 10:
73. 102 lbs. zinc, and 34 lbs. of tin: **74.** $\frac{3}{8}$:
75. A, -50; B, 200:
76. $2x+5y+3$: **77.** $2a^2b-3ab^2$: **78.** $3x^2-4x+5$:
79. $a^2x^2+ax^2-2ax$: **80.** ax^2-a^3x:
81. $2x^3-3x^2+x$: **82.** $5x^2-3xy-2y^2$:
83. $2ax-3by$: **84.** $\frac{10x}{x^3-2x^2-9x+18}$:
85. $\frac{x^3+23x^2-210}{x^3+2x^2-25x-50}$: **86.** $\frac{5y}{x-y}$:
87. $\frac{2a(a^2+32x^2)}{3(a^4-256x^4)}$: **88.** $\frac{3}{8(a-x)}$:

89. $\frac{96x^2}{27a^3 - 12ax^2 - 18a^2x + 8x^3}$: **90.** 10, and 5:

91. $1\frac{1}{4}$, and -2: **92.** $3a$, and $\frac{a}{2}$: **93.** 4, and -3:

94. $2\frac{1}{3}$, and -3: **95.** $\frac{12}{a}$, and $-\frac{10}{a}$: **96.** $\frac{5a}{2}$, and $-\frac{3a}{2}$:

97. 3, and -19: **98.** 5, and -3: **99.** 3, and -1:

100. 2, and $-\frac{1}{2}$: **101.** 4, and $\frac{8}{9}$: **102.** 7, and $\frac{2}{7}$:

103. 4, and $\frac{4}{3}$: **104.** $x = 17$ and 11, and $y = 11$ and 17:

105. $x = \pm 3$ and ± 7, and $y = \pm 7$ and ± 3:

106. $x = 9$, $y = 11$: **107.** $x = 10$, $y = 3$:

108. $x = \pm 5$ and ± 3, $y = \pm 3$ and ± 5:

109. 49, or -56: **110.** $\frac{2}{3}$, or $-\frac{35}{12}$: **111.** 17 by 12: **112.** 5.

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